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Quantum Mechanics and cosmology

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- Connections between classical and quantum theories.
- A classical dynamical system in a thermal bath.
- Hamiltonian mechanics and the time arrow.
- Harmonic oscillator in a thermal bath. The Planck constant h.
- Physics at the Planck scales. A chain of oscillators in a thermal bath.
- The lifetime of the quantum Universe and the cosmic dust.

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I Connections between classical and quantum theories

Hamiltonian mechanics:

1. Phase space (q,p)
2. Symplectic form $\omega^2(q,p)=\sum\limits_{k=1}^{n}\omega_k^{-1}\,dq_k\wedge dp_k$
3. The Hamilton function $H(q,p)$

The Piosson bracket for phase functions $f(q,p)$, $g(q,p)$: $\{f,g\}=\sum\omega_k(q,p)\{f,g/q_k,p_k\}$

where

$$\{f,g/q,p\}=\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right)=\frac{\partial(f,g)}{\partial(q,p)}$$

Equation of motion: $\dot f=\{f,H\}$

Hamiltonian mechanics contains all the essential ingredients of quantum mechanics

OPERATORS (B. Koopman, 1931) $\quad \dot f=\{H,\ /q,p\}f\equiv\hat H_{cl}f$

COMPLEX FUNCTIONS $\quad H=\frac{\omega}{2}(p^2+q^2)$ $\quad \boxed{\hat J^2=-1}$

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix}=\omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}\equiv\omega\hat J\begin{pmatrix}q\\ p\end{pmatrix}$$

Normal coordinates $z, \bar{z}$: $z = \frac{q + ip}{\sqrt{2}}$

$$\begin{pmatrix} \dot{z} \\ \dot{\bar{z}} \end{pmatrix} = -i\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}; \quad \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \equiv \hat{U} \begin{pmatrix} q \\ p \end{pmatrix};$$

$$\rightarrow \qquad \dot{z} = -i\omega z \qquad\qquad \hat{U}^{+}\hat{U} = 1$$

The imaginary unit $J \rightarrow i$ $\qquad \{\bar{z}, z\} = i$

Transformation $q, p \rightarrow z, \bar{z}$ is not canonical though $\hat{U}$ is unitary

PROBABILITIES complex ergodic systems are stochastic ones. Thus in deterministic theory there appear probabilities and the Gibbs distributions.

Hilbert space

The inner product $(f, g) = \frac{1}{Z} \int \bar{f} g e^{-\beta H(q,p)} d^n q \, d^n p$ for complex functions $\qquad f(q, p)$, $f(z)$

(B. Koopman, 1931, J. v. Neumann, 1932)

The Planck constant h

$$Z = \int e^{-\beta H(q,p)} d^n q \, d^n p$$

If Z exists, then the constant

$$Z^{1/n} = h$$

has the dimension of action.

$$\frac{h}{2\pi} = \frac{1}{\omega\beta} = \frac{kT}{\omega} \qquad \text{(oscillator)}$$

$$\beta\omega\bar{z}z = \bar{z}z/\hbar$$

$$H = \frac{\omega}{2}(p^2 + q^2) = \omega\bar{z}z$$

II A classical dynamical system in a thermal bath

Variations of canonical variables preserving the Gibbs distribution

$$\delta e^{-\beta H} = 0 \Rightarrow \delta H(q,p) = \sum \left(\frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p_i}\delta p_i\right) \equiv$$

$$\equiv \sum \nabla_i H \,\delta \vec{x}_i = 0, \quad \vec{x}_i = \vec{x}(q_i, p_i)$$

Solution: $\underbrace{\delta \vec{x}_i = \hat{J}_{ij} \nabla_j H \,\delta t}$ $\quad (\hat{J} - 2n \times 2n,\ \hat{J}^T = -\hat{J})$

In simplest case: $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial q_i$

Non-equilibrium distributions

$$\delta \vec{x} = \delta \vec{x}_\perp + \delta \vec{x}_\parallel$$

Measure $d\mu(\bar{z}, z) = \dfrac{d\bar{z} \wedge dz}{ih} e^{-\bar{z}z/\hbar}$ Harmonic oscil. $\hbar = \dfrac{1}{\beta\omega}$

Non-equil.: $d\mu_p = \dfrac{d\mu_p}{d\mu} d\mu \equiv P(\bar{z}, z)\, d\mu(\bar{z}, z)$

Let $\delta \vec{x}_\parallel \neq 0$: $H = \omega \bar{z} z \to \omega \bar{z} z + cz + \bar{c}\bar{z} + \ldots$

$$d\mu(\bar{z}, z) \to d\mu_f(\bar{z}, z) = |f(z)|^2 d\mu(\bar{z}, z)$$

$$f(z) = e^{-\beta c z}$$

PROJECTION

$R^2 \sin\theta \, d\varphi \wedge d\theta = d\mu(\bar{z}, z)$

$S^2(\theta, \varphi) \to z$ $\quad |z|^2 = \dfrac{1}{\beta} \ln \dfrac{1}{2\beta R^2 \sin^2 \theta/2}, \quad \arg z = \varphi$

III The time arrow (Eddington)

Classical equations of motion are invariant under transformation $t \to -t$, e.g., $\ddot{q} + \omega^2 q = 0$.

The corresponding Hamiltonian eqs. are not invariant: $\dot{q} = \omega p$, $\dot{p} = -\omega q$

Hamilt. dyns does not specify T-transformations of q, p.

NB Hamiltonian mechanics: phase space (PS), $H(q,p)$, $\underline{\underline{\omega(q,p)}}$:

$$\dot{f} = \omega(q,p)\left(\frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}\right).$$

This Eq. is invariant under $\begin{cases} t \to -t \\ \omega \to -\omega \end{cases}$

<u>Important</u> <u>PS is an oriented manifold</u>.

Transformation $\omega \to -\omega$ changes the orientation of PS

$H = \frac{\omega}{2}(p^2+q^2) = \omega\bar z z$, $z = \frac{q+ip}{\sqrt{2}}$

$\left.\begin{array}{l}\dot q = \omega p\\ \dot p = -\omega q\end{array}\right\} \longrightarrow \begin{array}{l}\boxed{\dot z = -i\omega z}\\ \dot{\bar z} = i\omega\bar z\end{array}$ $\{f,g/q,p\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$

$\{Q,P/q,p\} = 1$; $\{\bar z, z/q,p\} = \frac{1}{2}\{q-ip, q+ip\}_p = i$

(h) Heat bath: $\int dq\,dp\, e^{-\beta\frac{\omega}{2}(q^2+p^2)} \stackrel{def}{=} h$

$d\mu(\bar z, z) = \frac{d\bar z\wedge dz}{ih} e^{-\beta\omega\bar z z}$; $\frac{h}{2\pi} = \frac{1}{\beta\omega}$

(f(z)) $\vec X(q,p)$ — 2-vector. $\delta\vec X = \delta\vec X_\perp + \delta\vec X_\parallel$

$\delta e^{-\beta H} = 0 \longrightarrow \delta H = \nabla H\,\delta\vec X = 0$; $\nabla H\,\delta\vec X_\perp = 0$

$\delta\vec X_\perp = \hat J\,\nabla H\,\delta t$, $\hat J = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $\dot{\vec X} = J\nabla H$, $\dot q = \omega p$, $\dot p = -\omega q$

$\delta\vec X_\parallel$ (small) $H = \omega\bar z z \to \omega\bar z z + cz + \bar c\bar z + \cdots$

$d\mu(\bar z, z) \to d\mu_f(\bar z, z) = |f(z)|^2 d\mu(\bar z, z) \equiv$

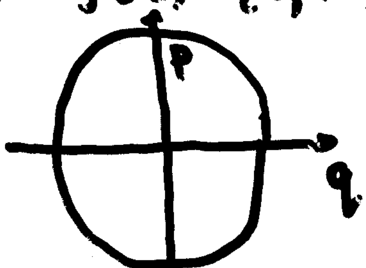
$\equiv P(\bar z, z)\,d\mu(\bar z, z)$; $f(z) = e^{cz}$

Evolution of non-equilibrium state

$\rho\,d\mu$, $\dot\rho = \{\rho, H\}_p$ $t_2 \gg \omega^{-1}$

HERE: (it is enough) consider $\dot f(z) = \{f, H\}_p =$

$= i\{f, H/\bar z, z\}$; it gives $\dot\rho$



IV Harmonic oscillator in a thermal bath

Evolution of non-equilibrium state is described by quantum mechanics

$$H=\frac{\omega}{2}\begin{pmatrix}q\\p\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix}=\frac{\omega}{2}\begin{pmatrix}z\\\bar z\end{pmatrix}\hat U^{*}\hat U^{+}\begin{pmatrix}z\\\bar z\end{pmatrix}=\frac{\omega}{2}\begin{pmatrix}z\\\bar z\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}z\\\bar z\end{pmatrix}$$

$H=\frac{\omega}{2}(z\bar z+\bar z z)$

$\boxed{t_z \gg \omega^{-1}}$ $\quad \dot f = i\{f, H/\bar z, z\} = -i\omega z\frac{d}{dz}f(z) \equiv \hat H_{cl} f,$

or $\quad (\partial_t+\omega\partial_\varphi)f(\varphi,t)=0 \qquad (\arg z=\varphi)$ $\quad z=|z|e^{i\varphi}$

Solution: $f(\varphi-\omega t)$ (wave)

SHOULD BE Periodic: $f(z(t))=\sum_{n=0}^{\infty}c_n Z_n(z);\ Z_n=\frac{z^n}{\sqrt{n!}}$

$Z_n(z)$: $\dot Z_n=-i\omega n Z_n$ (motion with frequences ωn)

Probability amplitudes $f(z)$: $\rightarrow$ DEFINE $(g,f)=\int\overline{g(z)}f(z)\,d\mu(\bar z,z)$

FOCK SPACE: functions of order $\rho\leq 2$

They describe non-equilibr. distributions

$Z_n(z/\sqrt{\hbar})$ form a basis: $(Z_n, Z_m)=\delta_{nm}$

From $(f,f)=\sum|c_n|^2=1\rightarrow|f(z)|^2=$ probability density

Commutation relations
$$\left.\begin{aligned}\hat{\bar z}\,\overline{g(z)}&=\bar z\,\overline{g(z)}\\ \hat z\,\overline{g(z)}&=\hbar\, d\bar g/d\bar z\end{aligned}\right\}\quad[\hat z,\hat{\bar z}]=\hbar$$

$$[\hat q,\hat p]=\frac{i}{2}[\hat{\bar z}+\hat z,\hat{\bar z}-\hat z]=i\hbar,\quad \hat H=\frac{\omega}{2}(\hat p^2+\hat q^2)=$$
$$=\frac{\omega}{2}(\hat z\hat{\bar z}+\hat{\bar z}\hat z)$$

The Schrödinger equation

Multiply: $i\hbar \,\Big|\, \dot{f} = -i\omega z \frac{d}{dz} f(z)$

$$i\hbar \dot{f}(z) = \hbar\omega z \frac{d}{dz} f(z) \equiv \hat{H}_{cl} f(z)$$

$$\hat{H}_{cl} = i\hbar \hat{H}_{cl} = \hbar\omega \hat{a}^+ \hat{a} \qquad \hat{a}^+ = z, \quad \hat{a} = \frac{d}{dz}$$

Spectrum of $\hat{H}_{cl}$: $\hbar\omega n$ (not $\hbar\omega(n+\frac{1}{2})$)

$E_n = \hbar\omega n + \frac{\hbar\omega}{2} \leftarrow$ the thermal contribution

From FOCK space to configuration space $\quad z \to q$ - not correct $[\exp(-\bar{z}z/\hbar)$ - not analitical$]$

$$\overline{Z_n(z)} \equiv \langle z | n \rangle, \quad \mathcal{H}_n(q) = \frac{1}{(2^n n! \sqrt{\pi})} H_n(q) e^{-q^2/2} \equiv \langle n | q \rangle$$

$$U(\bar{z}, q) = \sum_n \langle \bar{z} | n \rangle \langle n | q \rangle = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{\bar{z}^2 + q^2}{2} + \sqrt{2}\, q \bar{z}}$$

(V. Bargmann, 1961)

$$\int_{-\infty}^{\infty} dq \, U(\bar{z}, q) U(q, z) = e^{\bar{z}z}; \quad \int d\mu \, U(q, z) U(\bar{z}, q') = \delta(q - q')$$

$$\int_{-\infty}^{\infty} dq \, U(\bar{z}, q) \mathcal{H}_n(q) = Z_n(\bar{z}); \quad \int d\mu \, Z_n(z) U(\bar{z}, q) = \mathcal{H}_n(q)$$

$$\int d\mu(\bar{z}, z) \, d\mu(\bar{\zeta}, \zeta) \langle q | z \rangle (\bar{z} \zeta e^{\bar{z}\zeta}) \langle \bar{\zeta} | q' \rangle =$$

KERNEL

$$= \frac{1}{2}\left(-\frac{d^2}{dq^2} + q^2 - 1\right) \delta(q - q') \qquad (\omega = 1)$$

$$H^{cl}_{\bar{z}\zeta} = \bar{z} \zeta e^{\bar{z}\zeta}$$

$\tilde{V}$ A chain of oscillators in a heat bath

$$L=\frac{1}{2}\sum_n\left[m\dot{q}_n^2-\tilde{\gamma}(q_n-q_{n-1})^2-\gamma q_n^2\right] \qquad \boxed{l_P\sim 10^{-33}\,\text{cm}}$$

continuous limit: $L=\frac{1}{2}\int dx\,(\dot{\varphi}^2-\varphi'^2-M^2\varphi^2)$

$an\to x$, $q_n\sqrt{\frac{m}{a}}\to\varphi(x,t)$, $\frac{a^2\tilde{\gamma}}{m}\to 1$, $\frac{\gamma}{m}\to M^2$

$a\to 0,\ n\to\infty$

Klein-Fock: $(\Box-M^2)\varphi=0$ $\quad \cdots\underset{a}{\cdot\cdot}\cdots$

$$H=\frac{1}{2}\sum_n\left[\frac{p_n^2}{m}+\tilde{\gamma}(q_n-q_{n-1})^2+\gamma q_n^2\right]$$

Normal coordinates $u(k)$, $p(k)$

$$q_n=\int_{-\Delta}^{\Delta}dk\,u(k)\varphi_n^*(k),\quad p_n=\int_{-\Delta}^{\Delta}dk\,p(k)\varphi_n(k),\quad \varphi_n=\frac{1}{\sqrt{2\Delta}}e^{\frac{i\pi nk}{\Delta}}$$

$$H=\frac{1}{2}\int_{-\Delta}^{\Delta}dk\,\omega(k)\left(a^*(k)a(k)+a(k)a^*(k)\right)$$

$$\omega^2(k)=\frac{\gamma}{m}+\frac{4\tilde{\gamma}}{m}\sin^2\frac{\pi k}{2\Delta},\qquad u(k)=\frac{1}{\sqrt{2m\omega(k)}}\left(a^*(k)+a(-k)\right)$$

$$p(k)=i\sqrt{\frac{m\omega(k)}{2}}\left(a^*(k)-a(-k)\right)$$

Gibbs: $\prod_k da^*(k)\wedge da(k)\,e^{-\beta\int dk\,\omega(k)a^*(k)a(k)} =$

$$=\prod_k d\tilde{a}^*(k)\wedge d\tilde{a}(k)\lambda_k^{-1}e^{-\beta\omega\int dk\,\tilde{a}^*(k)\tilde{a}(k)}$$

$\tilde{a}(k)=\lambda_k^{1/2}a(k)$, $\lambda_k=\frac{\omega(k)}{\omega}$ $\qquad \hbar=\frac{1}{\beta\omega}$ FOCK

Thus: $(\Phi_1,\Phi_2)=\int\prod_k d\tilde{a}^*(k)\wedge d\tilde{a}(k)\,e^{-\int dk\frac{\tilde{a}^*(k)\tilde{a}(k)}{\hbar}}\,\overline{\Phi_1[\tilde{a}]}\,\Phi_2[\tilde{a}]$

VI The lifetime of the Universe

Restoration of equilibrium states

$$\ddot{q}(t) + \omega^2 q(t) = 0 \rightarrow \ddot{q} + \alpha\dot{q} + \omega^2 q = 0$$

$\alpha > 0$ specifies friction. For infinitesimal α solution is

$$q(t) = c_1 e^{-i(\omega - i\frac{\alpha}{2})t} + c_2 e^{i(\omega + i\frac{\alpha}{2})t}$$

Classical deterministic motion disappears when $t \rightarrow \infty$ ($t > t_r \sim \frac{1}{\alpha}$)

Any coherent motion dies away

Photons „grow old" (factor $e^{-\alpha\frac{t}{2}}$)

It looks as decreasing of the photon number with time, i.e. as absorbtion of photons by space (from distant sources). In a way it imitates cosmic dust.

<u>Difference</u>: „the absorbtion" is the same for all frequences.

19.04.04

QM & Cosmology

The title is a little bit deceiving. One may imagine that I'll speak about cosmological consequences following from quantum nature of microworld ~~in cosmology~~. In fact I'll speak about origin of quantum theory. I believe to uncover it one should turn to Planck scales. It turns out that one can construct a model demonstrating appearance of quantum description in framework of pure classical theory (an oscillator in a thermal bath). There follow some quite unexpected consequences for the Universe.

1. CONNECTIONS BETWEEN CLASSICAL AND QUANTUM THEORIES

The time arrow

Classical equations of motion are invariant under transformation $t \to -t$

e.g. $\ddot{q} + \omega^2 q = 0$ $\quad \frac{d^2 q(t)}{dt^2} + \omega^2 q(t) = 0$

The corresponding Hamiltonian equations ~~are~~

$$\left.\begin{aligned} \frac{dq}{dt} &= \omega p \\ \frac{dp}{dt} &= -\omega q \end{aligned}\right\}$$

are not invariant because Hamilt. dyn. does not specify T-transformations of q, p.

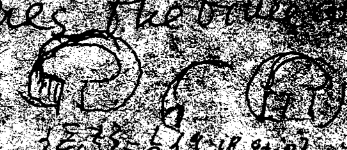
Notice. To define Ham. mech one has to define both $H(q,p)$ and $\omega(q,p)$ and $\frac{df}{dt} = \omega(q,p)\left(\frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}\right)$

The Eq is invariant under tr-n $\begin{cases} t \to -t \\ \omega \to -\omega \end{cases}$

Thus, it is the symplectic form which defines the time arrow.

Besides, the PS is an oriented manifold.

In fact, the orientation of PS defines the [illegible]



Об условии $\delta H(q,p) = 0$ $(\delta f(\vec{x}) = 0)$

$\delta f(\vec{x}) = \nabla f\,\delta\vec{x} = 0 \Rightarrow \nabla f \perp \delta\vec{x}$ (*)

ТЕОРЕМА: Условие (*) необходимо и достаточно для экстремальности $S = \int (\vec{x} J \dot{\vec{x}} - f(\vec{x}))\,dt$, $\delta x(t_1) = \delta x(t_2) = 0$. (**)

(1) Необходимость. Из (**) следует, что $J\dot{\vec{x}} = \nabla f$, $J^T = -J$, $J^2 = -1$, и т.е. $\delta\vec{x} = J\nabla f\,\delta t$. Т.е. из (**) следует, что $\delta\vec{x}$ [illegible] вращении так, чтобы [illegible] (*)

(2) Достаточность. Из δx следует, [illegible]

$$\delta\vec{x}(t) = \lambda(t) J \nabla f(\vec{x})\,\delta t, \quad \nabla f J \nabla f = 0.$$

[illegible]

$\delta\vec{x}(J\dot{\vec{x}} - \nabla f)$ [следует из: $\delta\vec{x}(\dot{\vec{x}} + J\nabla f) = 0$],

получаем $\delta \int_{t_1}^{t_2} (\vec{x} J \dot{\vec{x}} - f(\vec{x}))\,dt = 0$,

т.е. вариационный принцип в форме Гамильтона.

For our purposes it is enough to take [illegible] the classical equations of motion conserve the Gibbs distribution. Ergo: among $\delta\vec{x}_1$ there $\delta x_1^{classic}$, i.e. those following from classical equations of motion.

НЕ доказано НЕТ!!