THREE-BODY RESONANCES*

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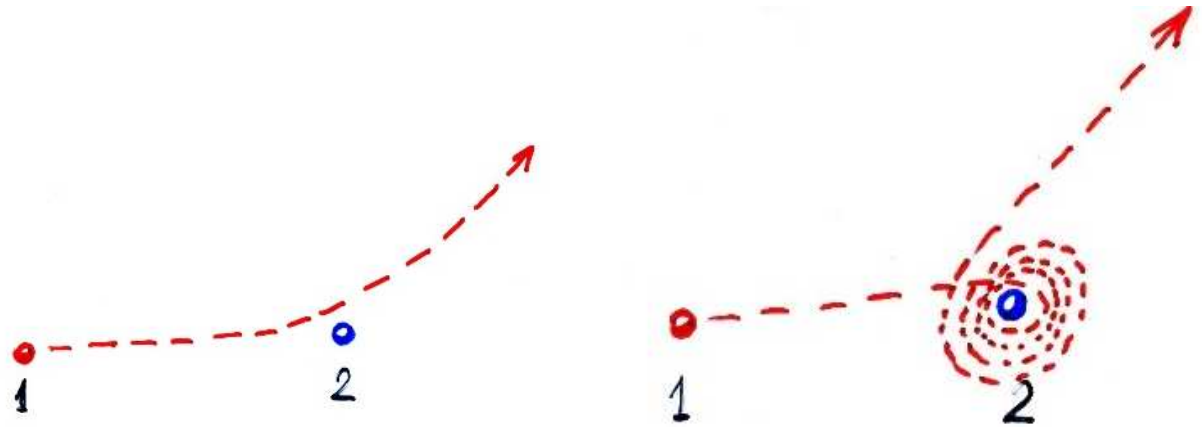
1 Introduction

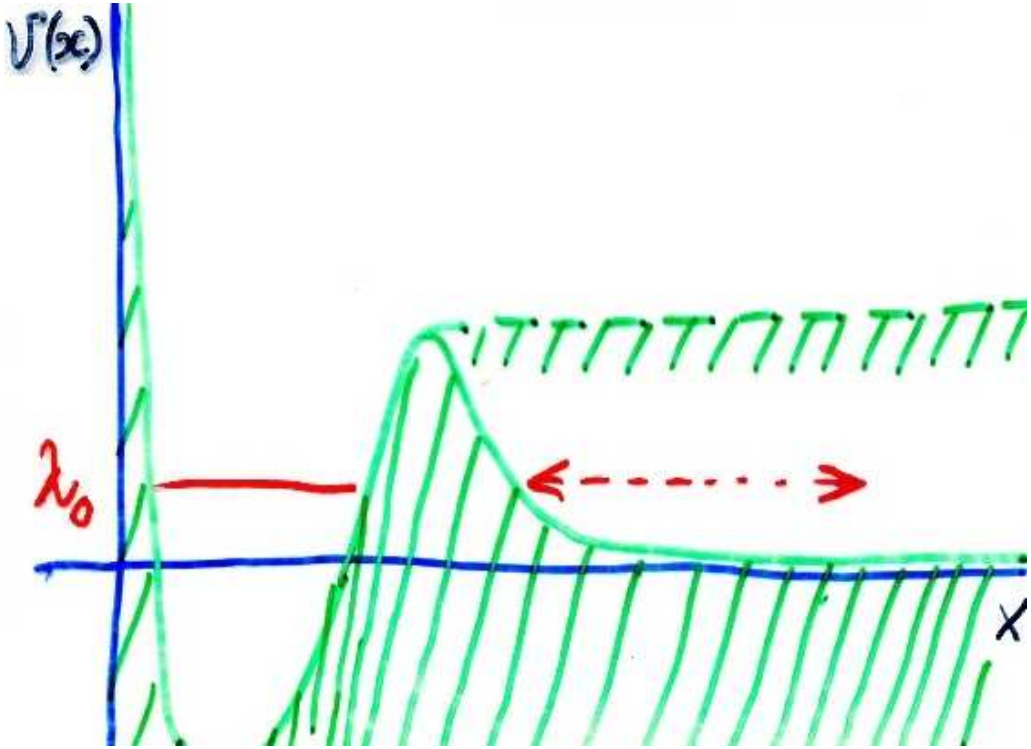
Resonance is one of the most interesting and intriguing phenomena in quantum scattering. With a resonance one usually associates an unstable (metastable) state that only exists during a certain time.

Typical example: two-particle scattering.

$$\mathrm{i}\frac{\partial \psi}{\partial t} = (-\Delta + V)\psi$$



Particle may “live” together during a certain period forming an unstable (“resonant”) state.



A barrier resonance

In the following:

Resonance (in a narrow sense) $\Longleftrightarrow$ a specific complex energy, the energy of a resonant state

$$z = E + \mathrm{i}\frac{\Gamma}{2}.$$

On the history of the subject

S-matrix interpretation

Gamov (1928): resonances $\Longleftrightarrow$ poles of the scattering amplitudes
[α decay of heavy nuclei] (that is, those of the S-matrix)

Titchmarsh (1946): Resonances are also poles
of the continued resolvent kernel

Jost functions (1940's)

Complex scaling approach

Lovelace ($\sim$ 1964) "complex rotation" of Hamiltonian
Balslev, Combes (1971) (rotation of the continuous spectrum)
Simon, Hagedorn, Hunziker,...

Hagedorn (1979): for a wide class of potentials the scaling resonances are also the scattering matrix resonances.

We also mention the Lax-Phillips approach and various versions of perturbation theory for resonances (Albeverio, Livšic, Howland, Rauch, ...)

In contrast to the "normal" bound and scattering states, from mathematical point of view the resonant ones are still a mysterious subject.

Many questions still remain unanswered.

In particular, how to get characteristics for scattering of a particle on a resonant state of two other particles?

There is a problem even with the definition of resonance:

> Resonances are NOT a unitary invariant of a self-adjoint (Hermitian) operator

B. Simon [J. Chem. Phys. **14** (1978)]: Always an extra structure is necessary to describe a resonance. Say, an "unperturbed dynamics" (in quantum scattering theory); geometry (in acoustical or optical problems).

> ***Resonances are always relative as the scattering matrix itself***.

Typical setup

Kinetic energy operator $H_0 \Longleftrightarrow$ unperturbed dynamics

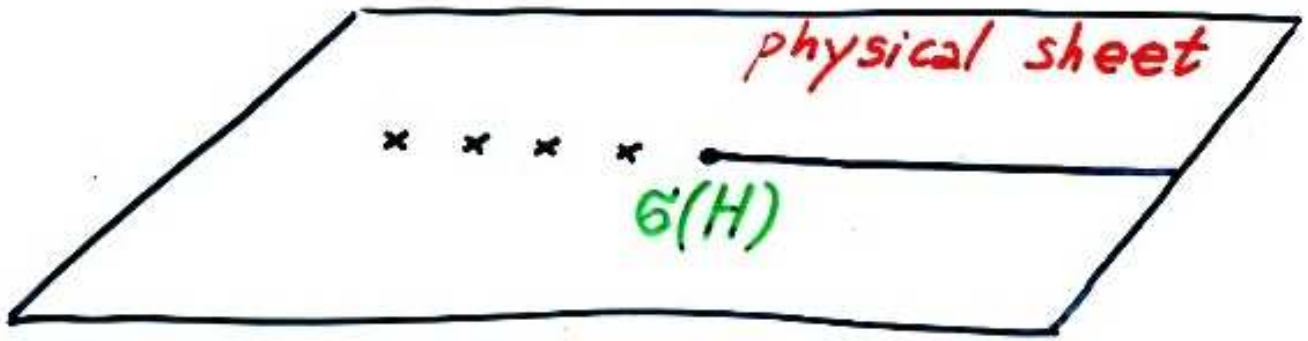
$$H = H_0 + V, \qquad V \quad \text{interaction.}$$

The resolvent

$$R(z) = (H - z)^{-1}$$

is an analytic operator-valued function of $z \in \mathbb{C} \setminus \sigma(H)$.

The spectrum $\sigma(H)$ is a natural boundary for holomorphy domain of $R(z)$ considered as an operator-valued function.



However the kernel $R(\cdot,\cdot,z)$ (given in some specific representation) may admit analytic continuation through the continuous spectrum $\sigma_c(H)$.

Or the form $\langle R(z)\varphi,\psi\rangle$ does this for any φ,ψ of a dense subset of the Hilbert space $\mathscr{H}$.

Or the "compressed" resolvent $PR(z)P$ admits such a continuation for P the orthogonal projection onto a subspace of $\mathscr{H}$.

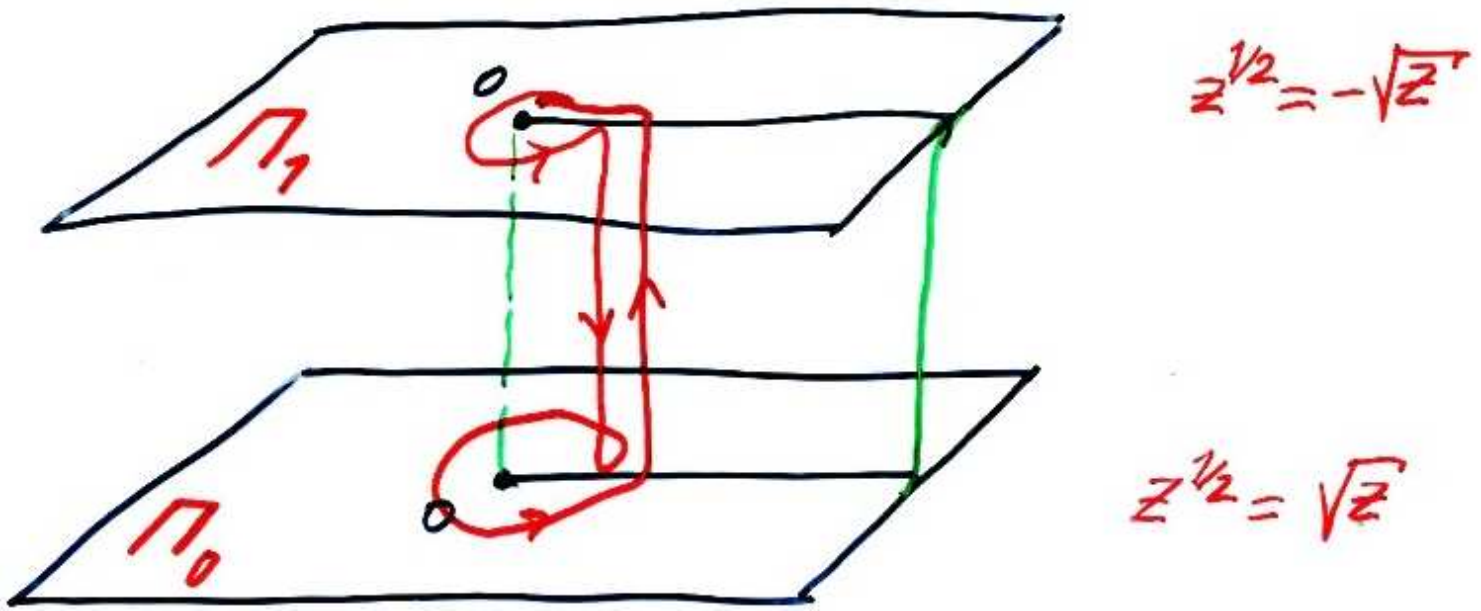
In any case one deals with the Riemann surface of an analytical function.

Simplest example: $H = H_0 = -\Delta$, the two-body kinetic energy operator. In this case

$$R_0(\boldsymbol{x}, \boldsymbol{x}', z) = \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}z^{1/2}|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|},$$

where $\boldsymbol{x}, \boldsymbol{x}'$ are three-dimensional vectors.

Two-sheeted Riemann surface (coincides with that of $z^{1/2}$).



Analytic function is uniquely defined by its values on a set in $\mathbb{C}$ having limiting point(s). Thus, if one knows the resolvent $R(z)$ (or T-matrix, S-matrix) on the physical sheet then one may, in principle, to express it on unphysical sheets through its values in the physical sheet.

In such a case all the study of resonances would reduce to a work completely on the physical sheet!

We have suggested just such representations: ***Explicit representations for*** $R(z)$, $T(z)$, ***and*** $S(z)$ ***on unphysical sheets in terms of these quantities themselves taken from the physical sheet.***

In particular, ***these representations show which blocks of the scattering matrix are "responsible" for resonances on a certain unphysical sheet.***

References for this part of the material:

AM, Theor. Math. Phys. **97**, 692 (1993)
AM, Theor. Math. Phys. **107**, 784 (1996)
AM, Mathematische Nachrichten **187**, 147 (1997)
AM and E. Kolganova, Few-Body Syst. Suppl. **10**, 75 (1999)
AM, Few-Body Syst. **38**, 115 (2006)

2 An explicitly solvable model with a resonance

We start considering the resonances *ab ovo*. Namely, we first study a simplest model Hamiltonian where a resonance is computed explicitly. For this resonance it is even possible to prove, quite easy, that the corresponding resonance state decays according to the exponential law.

If you are already quite experienced with resonances, you may simply skip this section and go directly to Section 3 which is devoted to the explicit representations for the T-matrix, scattering matrix, and resolvent on unphysical energy sheet in the two-body problem.

Consider a two-channel Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ consisting of an infinite-dimensional Hilbert space $\mathscr{H}_0$ (channel 0) and a one-dimensional space $\mathscr{H}_1 = \mathbb{C}$ (channel 1). The elements of $\mathscr{H}$ are represented as vectors $u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ where $u_0 \in \mathscr{H}_0$ and $u_1 \in \mathscr{H}_1$, with u_1 being a complex number. The inner product $\langle u, v \rangle_{\mathscr{H}} = \langle u_0, v_0 \rangle + u_1 \bar{v}_1$ in $\mathscr{H}$ is naturally defined via the inner products $\langle u_0, v_0 \rangle$ in $\mathscr{H}_0$ and $u_1 \bar{v}_1$ in $\mathscr{H}_1$.

As a Hamiltonian in $\mathscr{H}$ we consider the 2×2 operator matrix

$$H = \begin{pmatrix} h_0 & b \\ \langle \cdot, b \rangle & \lambda \end{pmatrix} \tag{2.1}$$

where h_0 is the (selfadjoint, that is, Hermitian) operator in $\mathscr{H}_0$, and λ a real number. A vector $b \in \mathscr{H}_0$ provides the coupling between the channels.

We mention that the Hamiltonian (2.1) resembles one of the celebrated Friedrichs models (see [K. O. Friedrichs, Comm. Pure Appl. Math. **1** (1948), 361]; some more details on Hamiltonians like (2.1) can be found in [R. Mennicken and AM, Math. Nachr. **201** (1999), 117] and [AM, W. Sandhas, and V. B. Belyaev, J. Math. Phys. **42** (2001), 2490]).

- For example, we can assume that h_0 is the kinetic energy operator of a particle on a straight line,

$$h_0 = -\frac{d^2}{dx^2}, \quad x \in \mathbb{R}. \tag{2.2}$$

In this case $b = b(x)$ is a usual square-integrable function of the variable x. Notice, that the resolvent of the operator (2.2) has the following kernel (as usually, it is called the free Green function):

$$r_0(x, x', z) = -\frac{e^{iz^{1/2}|x-x'|}}{2iz^{1/2}}, \tag{2.3}$$

where $z^{1/2}$ is understood as a two-sheeted function in the sense explained above on page 9.

If there is no coupling between the channels 0 and 1, i. e. for $b = 0$, then the spectrum of H consists of the spectrum of h_0 and the additional eigenvalue λ. We assume that the continuous spectrum $\sigma_c(h_0)$ of the Hamiltonian h_0 is not empty and that the eigenvalue λ is embedded into $\sigma_c(h_0)$. For now it is also assumed that λ is not a threshold point of $\sigma_c(h_0)$, and that the spectrum $\sigma_c(h_0)$ is absolutely continuous in a sufficiently wide neighborhood of λ.

A nontrivial coupling $(b \neq 0)$ between the channels 0 and 1 will, in general, shift the eigenvalue λ to an unphysical sheet of the energy plane. The resulting perturbed energy appears as a resonance, i. e., as a pole of the analytic (or, more precisely, meromorphic) continuation of the resolvent $R(z) = (H - z)^{-1}$ taken between suitable states. We suppose that such a continuation through the absolutely continuous spectrum of h_0 in some neighborhood of λ is possible at least for the matrix element $\langle r_0(z)b, b\rangle$ of the resolvent $r_0(z) = (h_0 - z)^{-1}$. This yields a meromorphic continuability at least for the compressed resolvent $P_1(H - z)^{-1}\big|_{\mathscr{H}_1}$ where P_1 denotes the orthogonal projection onto the (one-dimensional) space $\mathscr{H}_1$.

Actually, the explicit representation for the resolvent $R(z)$ is easily seen to be

$$R(z) = \begin{pmatrix} r_0(z) + \dfrac{r_0(z)b\langle\,\cdot\,,b\rangle r_0(z)}{M(z)} & -\dfrac{r_0(z)b}{M(z)} \\ -\dfrac{\langle\,\cdot\,,b\rangle r_0(z)}{M(z)} & \dfrac{1}{M(z)} \end{pmatrix}. \tag{2.4}$$

where the scalar function $M(z)$ reads $M(z) = \lambda - z - \langle r_0(z)b, b\rangle$. Thus, if the function $\langle r_0(z)b, b\rangle$ admits analytic continuation in z through an interval of the absolutely continuous spectrum $\sigma_c(h_0)$ of the entry h_0, then the function $P_1(H-z)^{-1}\big|_{\mathscr{H}_1} = M^{-1}(z)$ admits such a continuation, too.

From (2.4) it is obvious that the poles of $R(z)$ on the physical sheet are either due to zeros of the function $M(z)$ or due to poles of the resolvent $r_0(z)$. The latter correspond to the discrete spectrum of the operator h_0 which, thus, may generate a part of the point spectrum of H.

In any case it is clear that the perturbation of the eigenvalue λ only corresponds to solutions of the equation $M(z) = 0$, i. e. to solutions of

$$z = \lambda - \langle r_0(z)b, b\rangle. \tag{2.5}$$

Equation (2.5) has no roots z with $\mathrm{Im}\, z \neq 0$ on the physical sheet. This is clear since for being eigenvalues of the selfadjoint operator H, they have, of course, to be real. Thus, Eq. (2.5) may have solutions only on the real axis and in the unphysical sheet(s) of the Riemann surface of the resolvent $r_0(z)$.

Assume, in addition, that the channel Hamiltonian h_0 generates no resonances close to λ in a domain $\mathscr{D}$ of the unphysical sheet which is neighboring the physical sheet from below the cut. This assumption implies that for a wide set of unit vectors $\widehat{b} = b/\|b\|$ the quadratic form $\langle r_0(z)b, b\rangle = \|b\|^2\langle r_0(z)\widehat{b}, \widehat{b}\rangle$ can be analytically continued in $\mathscr{D}$. Moreover, under certain smallness conditions for $\|b\|$, equation (2.5) is uniquely solvable and its solution z_{res}^- in the lower complex half-plane reads

$$z_{\text{res}}^- \underset{\|b\|\to 0}{=} \lambda - \langle r_0(\lambda + \mathrm{i}0)b, b\rangle + o(\|b\|^2). \tag{2.6}$$

The real and imaginary parts of the resonance $z_{\text{res}}^- = E - \mathrm{i}\dfrac{\Gamma}{2}$, thus, are given by

$$\begin{aligned} E &= \lambda - \operatorname{Re}\langle r_0(\lambda + \mathrm{i}0)b, b\rangle + o(\|b\|^2), \\ \Gamma &= 2\operatorname{Im}\langle r_0(\lambda + \mathrm{i}0)b, b\rangle + o(\|b\|^2). \end{aligned} \tag{2.7}$$

- Notice that there is a particular case of the model (2.1) which is completely solvable but somewhat distinct from the assumptions that lead us to the approximate solution (2.6). In this particular case we take λ just on a threshold of $\sigma_c(h_0)$. Namely, let h_0 be given by (2.2) and let $\lambda = 0$ and $b(x) = \beta\delta(x)$, $0 \neq \beta \in \mathbb{R}$. Then by (2.3) equation (2.5) is reduced to

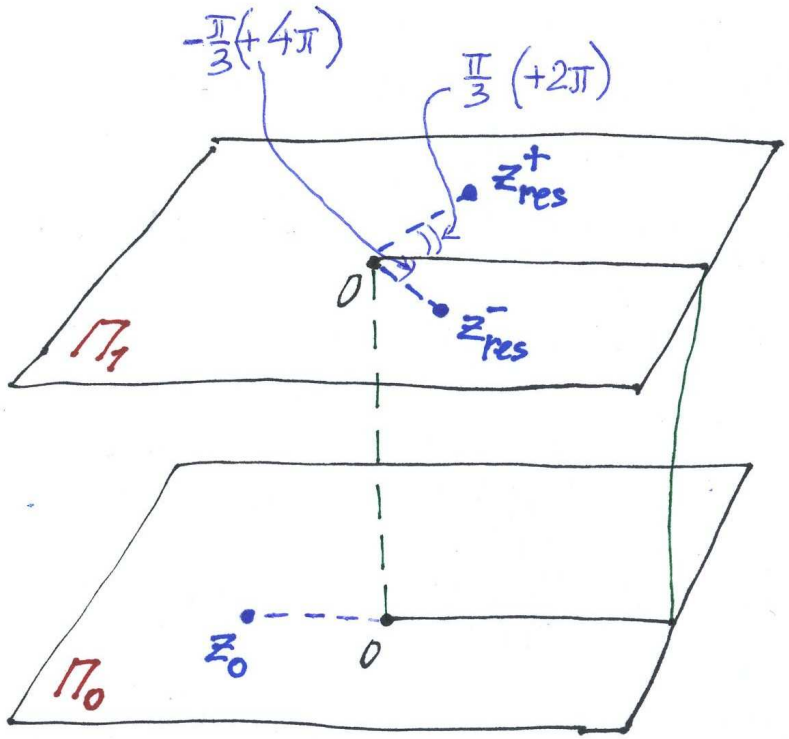
$$z = -\frac{\mathrm{i}}{2}\frac{\beta^2}{z^{1/2}}.$$

By inspection, this equation has exactly three solutions:

(1) binding energy $z_0 = -(\beta^2/2)^{2/3}$
(in the physical sheet),

(2) resonance $z_{\mathrm{res}}^{+} = (\beta^2/2)^{2/3}\mathrm{e}^{\mathrm{i}(2\pi+\pi/3)}$
(in the upper half-plane of the unphysical sheet),

(3) resonance $z_{\mathrm{res}}^{-} = (\beta^2/2)^{2/3}\mathrm{e}^{\mathrm{i}(4\pi-\pi/3)}$
(in the lower half-plane of the unphysical sheet).



The fact that these binding energy and resonances are proportional not to β^2 (as one could expect looking at formula (2.6)) but to $\beta^{4/3}$ corresponds just to the threshold position of the embedded eigenvalue $\lambda = 0$ with respect to the continuous spectrum $\sigma_c(h_0) = \sigma(h_0) = [0, +\infty)$.

Exponential decay of the resonance state

Let us suppose that an initial state of the system described by the Hamiltonian (2.1) corresponds exactly to the "wave function" $\varphi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then evolution of the system in time is described by the solution $\psi(t)$ of the Cauchy problem

$$i\frac{d\psi}{dt} = H\psi, \qquad \psi\big|_{t=0} = \varphi, \tag{2.8}$$

that is,

$$\psi(t) = \mathrm{e}^{-\mathrm{i}Ht}\varphi.$$

The probability $P_\varphi(t)$ to find the system at the time t still in the state φ is given by

$$P_\varphi(t) = |\langle \psi(t), \varphi \rangle|^2.$$

The remainder $1 - P_\varphi(t)$, hence, determines the probability for the state φ to decay into open channels of the continuous spectrum of the sub-Hamiltonian h_0.

To estimate the probability $P_\varphi(t)$, we use the standard integral representation of a function of an operator via its resolvent. In the case considered this means

$$\exp\{-\mathrm{i}Ht\} = -\frac{1}{2\pi\mathrm{i}}\oint\limits_{\gamma} dz\,\mathrm{e}^{-\mathrm{i}zt}(H-z)^{-1}. \tag{2.9}$$

The integration in (2.9) is performed in the physical sheet along a contour γ encircling counterclockwise the spectrum of the matrix H. Recall that, due to the selfadjointness of the operator H, this spectrum is real. Taking into account representations (2.4) and (2.9) one finds

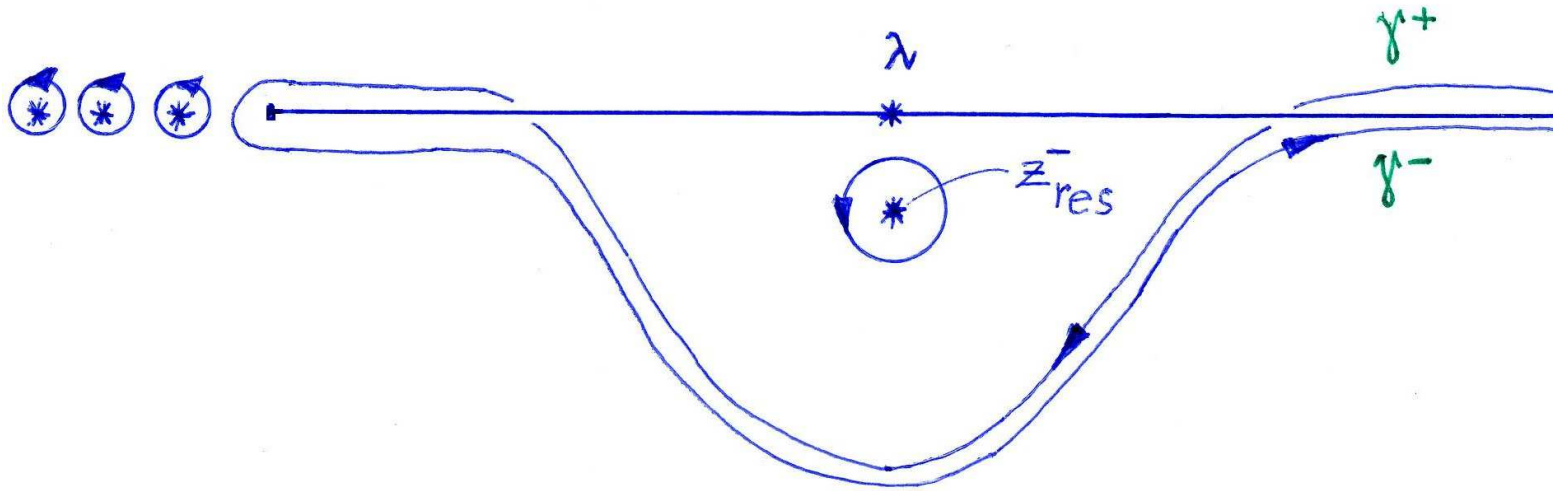
$$\langle\psi(t),\varphi\rangle = -\frac{1}{2\pi\mathrm{i}}\oint\limits_{\gamma} dz\,\frac{\exp(-\mathrm{i}zt)}{\lambda - z - \langle r_0(z)b,b\rangle} \tag{2.10}$$

Under the assumption that b is very small this leads to the following result: The behavior of the integral (2.10) for $t>0$ is described by the formula

$$\langle\psi(t),\varphi\rangle = \mathrm{e}^{-\mathrm{i}z_{\mathrm{res}}^{-}t}\left[1-O(\|b\|^2)\right]+\varepsilon(t) \tag{2.11}$$

where the background term $\varepsilon(t)=O(\|b\|^2)$ is small, $|\varepsilon(t)|\ll 1$, for all $t>0$.

The proof of the asymptotics (2.11) is performed by estimating the contribution of the resonance pole $z^-_{\rm res}$ to the integral (2.10). This is done by deforming parts of the contour γ situated in a neighborhood of the energy λ (see the picture below).



A scheme showing the deformation of the integration path γ. The part γ^+ of the resulting contour lying in the lower half-plane belongs to the unphysical sheet, while the part γ^- lies completely in the physical sheet. The asterisks "$*$" on the l.h.s. part of the picture denote the (possible) discrete eigenvalues of the Hamiltonian H while the solid straight line corresponds to the continuous spectrum.

A part γ^+ of γ, situated initially on the upper rim of the cut, is shifted into the neighboring unphysical sheet. Having done such a deformation one finds explicitly the residue of the integrand in (2.10) at $z = z_{\text{res}}^-$.

An analogous deformation of a part γ^- of γ, situated initially on the lower rim, is performed in a domain $\operatorname{Im} z < 0$ of the physical sheet.

(!) It is assumed that, though the parts γ^+ and γ^- belong to different energy sheets, their positions on these sheets coincide.

It is also assumed that b is so small and that the deformed contour γ lies so far from λ that for any $z \in \gamma^\pm$ the estimate

$$|\langle r_0(z)b, b\rangle| \ll |\lambda - z| \tag{2.12}$$

holds.

Thus, the integration in (2.10) around the continuous spectrum of H, except the residue at $z = z^-_{\rm res}$, gives

$$-\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma^+} dz \exp(-\mathrm{i}zt) \left(\frac{1}{\lambda - z - \langle r_0(z)b, b\rangle^+} - \frac{1}{\lambda - z - \langle r_0(z)b, b\rangle^-} \right)$$
$$= -\frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma^+} dz \exp(-\mathrm{i}zt) \, \frac{\langle r_0(z)b, b\rangle^+ - \langle r_0(z)b, b\rangle^-}{[\lambda - z - \langle r_0(z)b, b\rangle^+]\,[\lambda - z - \langle r_0(z)b, b\rangle^-]}. \tag{2.13}$$

Here we use a specific notation $\langle r_0(z)b, b\rangle^+$ for the values of the function $\langle r_0(z)b, b\rangle$ at points z belonging to the curve γ^+ (i. e., lying in the unphysical sheet), and $\langle r_0(z)b, b\rangle^-$ for the values of $\langle r_0(z)b, b\rangle$ at the same points of the curve γ^- (i. e., lying in the physical sheet).

Both $\langle r_0(z)b, b\rangle^\pm$ are of the order of $O(\|b\|^2)$, and by the earlier assumption (2.12) we have $|\langle r_0(z)b, b\rangle^\pm| \ll |\lambda - z|$, while the exponential $\exp(-\mathrm{i}zt)$ at $\mathrm{Im}\, z < 0$ is decreasing for $t > 0$.

The value of the function (2.13), thus, is always small, having an order of $O(\|b\|^2)$, and is even decreasing (in general nonexponentially) with increasing t. We include the contribution of this function into the background term $\varepsilon(t)$.

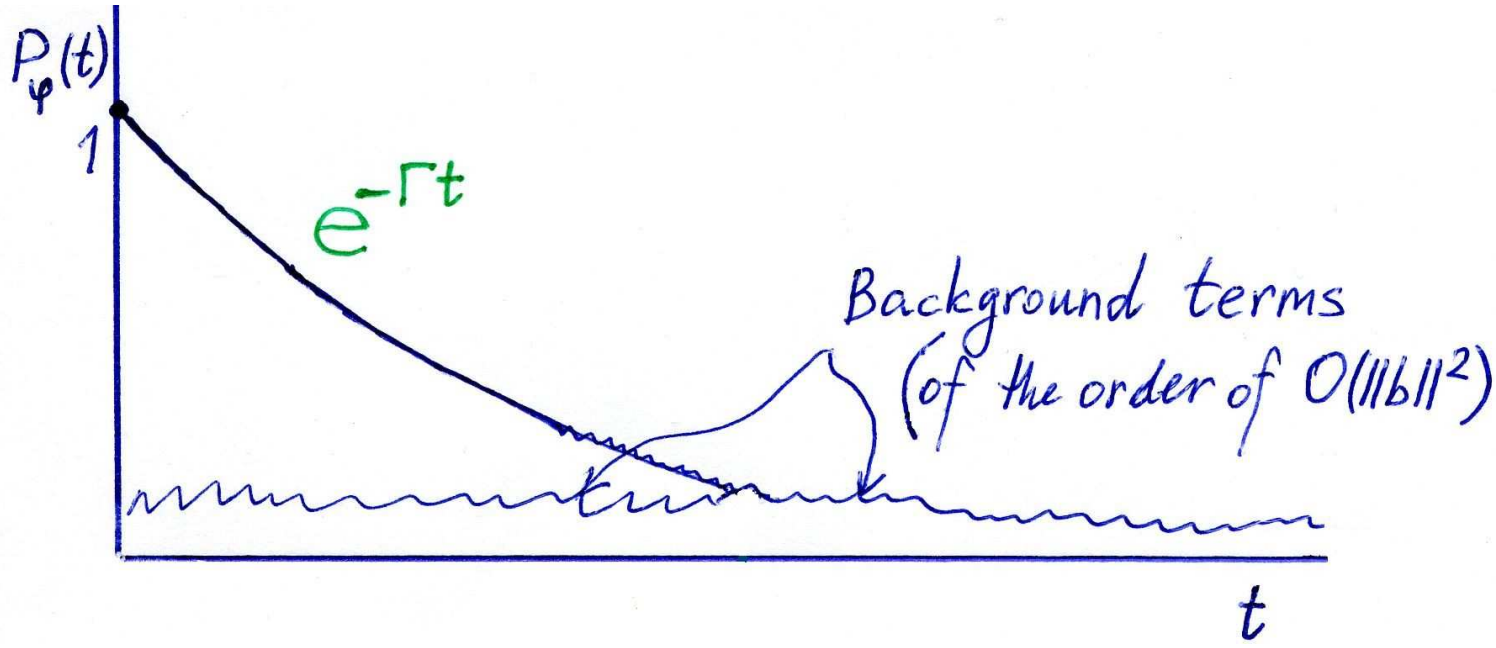
The term $\varepsilon(t)$ also includes a contribution to (2.10) from the residues at the discrete eigenvalues of H. Apart from factors oscillating when t changes, the value of this contribution remains practically the same for all $t \geq 0$.

Formula (2.11) shows explicitly that in a large time interval $0 \leq t < T$,

$$T \sim \frac{\left|\log\left(\max|\varepsilon(t)|\right)\right|}{\Gamma},$$

the decay of the particular state φ we have "prepared" is actually of an exponential character. The rate of this decay is determined mainly by the width Γ of the resonance z_{res}^{-}, namely

$$P_\varphi(t) \cong \exp\{-\Gamma t\}\left(1 + O(\|b\|^2)\right) + O(\|b\|^2). \tag{2.14}$$

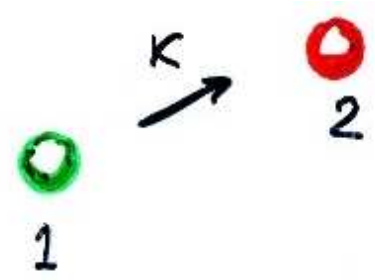


Behavior of the survival probability $P_\varphi(t)$ $\big($according to formula (2.14)$\big)$.

Thus, we have shown that, after turning the coupling between the channels 0 and 1 on, the state $\varphi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ decays according to the exponential law, at least in the beginning.

Such a situation is quite common for resonance states. Exponential decay is observed in various physical experiments.

3 Two-Body Problem



$$\boldsymbol{k} = \left[\frac{\mathrm{m}_1+\mathrm{m}_2}{2\mathrm{m}_1\mathrm{m}_2}\right]^{1/2} \cdot \frac{\mathrm{m}_1\mathbf{p}_2 - \mathrm{m}_2\mathbf{p}_1}{\mathrm{m}_1+\mathrm{m}_2} \qquad \text{(reduced relative momentum)}$$

$$(hf)(\boldsymbol{k}) = \boldsymbol{k}^2 f(\boldsymbol{k}) + (Vf)(\boldsymbol{k})$$

$V(\boldsymbol{k},\boldsymbol{k}') = V(\boldsymbol{k}-\boldsymbol{k}')$ — in case of local potentials —
and $V(\boldsymbol{k}) = \overline{V(-\boldsymbol{k})}$, $\boldsymbol{k} \in \mathbb{R}^3$.

For simplicity we assume that $V(\boldsymbol{k})$ is holomorphic for all $\boldsymbol{k} \in \mathbb{C}^3$.

(In fact it suffices to require the holomorphy of $V(\boldsymbol{k})$ only in a "strip" $|\mathrm{Im}\,\boldsymbol{k}| < a$ for some $a > 0$.)

Resolvents:

$$r_0(z) = (h_0 - z)^{-1}, \qquad (h_0 f)(\boldsymbol{k}) = \boldsymbol{k}^2 f(\boldsymbol{k}),$$
$$r(z) = (h - z)^{-1}.$$

$$r_0(\boldsymbol{k}, \boldsymbol{k}', z) = \frac{\delta(\boldsymbol{k} - \boldsymbol{k}')}{\boldsymbol{k}^2 - z}$$

T-operator:

$$t(z) = V - V r(z) V \quad \Longrightarrow \quad r(z) = r_0(z) - r_0(z) t(z) r_0(z)$$

The Lippmann-Schwinger equation for $t(z)$

$$t(z) = V - V r_0(z) t(z),$$

that is

$$t(\boldsymbol{k}, \boldsymbol{k}', z) = V(\boldsymbol{k}, \boldsymbol{k}') - \int_{\mathbb{R}^3} d\boldsymbol{q} \frac{V(\boldsymbol{k}, \boldsymbol{q}) t(\boldsymbol{q}, \boldsymbol{k}', z)}{\boldsymbol{q}^2 - z}. \tag{3.1}$$

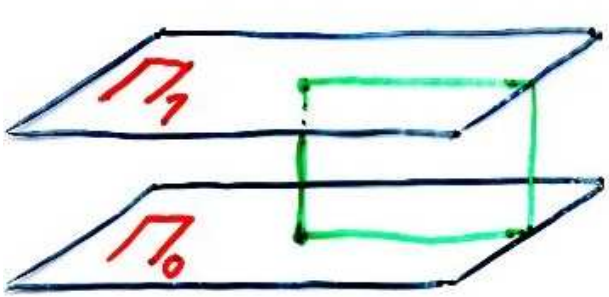
Clearly, all the dependence of t on z in (3.1) is determined by the integral term on the r.h.s. part. This integral is nothing but a particular case of the Cauchy type integral

$$\Phi(z) = \int_{\mathbb{R}^N} d\boldsymbol{q} \frac{f(\boldsymbol{q})}{\lambda + \boldsymbol{q}^2 - z} \qquad (N = 3 \text{ or, later on, } N = 6 \text{ in the Faddeev eqs.}).$$

Cauchy-type integrals of just the same form are also present in the Faddeev equations.

Denote by $\mathfrak{R}_\lambda$ the Riemann surface of the function

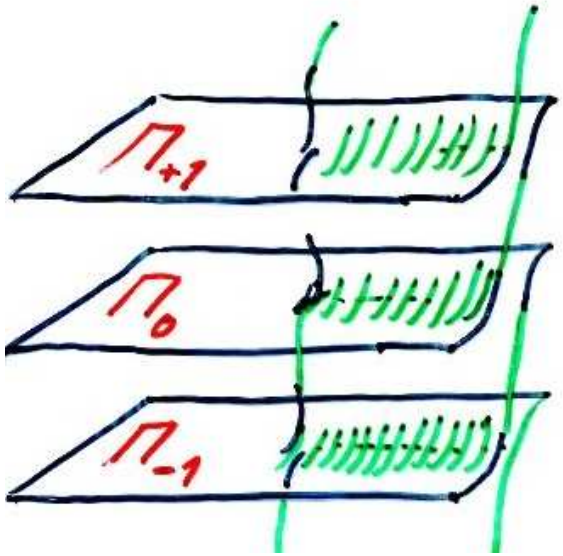
$$\zeta(z) = \begin{cases} (z-\lambda)^{1/2}, & N \quad \text{odd}, \\ \ln(z), & N \quad \text{even}. \end{cases}$$



$\ell = 1 \qquad (z-\lambda)^{1/2} = -\sqrt{z-\lambda}$

$\ell = 0 \qquad (z-\lambda)^{1/2} = \sqrt{z-\lambda}$

N odd



$\ell = +1$

$\ell = 0$

$\ell = -1$

$$\ln(z-\lambda) = \ln|z-\lambda| + i\varphi + 2\pi i\ell$$

N even

Lemma 1. *For a holomorphic $f(\boldsymbol{q})$, $\boldsymbol{q} \in \mathbb{C}^N$, the function*

$$\Phi(z) = \int_{\mathbb{R}^N} d\boldsymbol{q} \frac{f(\boldsymbol{q})}{\boldsymbol{\lambda} + \boldsymbol{q}^2 - z}$$

is holomorphic on $\mathbb{C} \setminus [\boldsymbol{\lambda}, +\infty)$ and admits the analytic continuation onto $\mathfrak{R}_\lambda$ as follows

$$\Phi(z)|_{\Pi_l} = \Phi(z) - l\,\pi \mathrm{i} (\sqrt{z-\boldsymbol{\lambda}})^{N-2} \int_{S^{N-1}} d\widehat{\boldsymbol{q}} f(\sqrt{z-\boldsymbol{\lambda}}\widehat{\boldsymbol{q}}). \qquad (3.2)$$

Do not confuse l with orbital moment! Here l is the first letter of the Russian word "*list*" $\Leftrightarrow$ "*sheet*".

We only deal with $N = 3$ in the two-body problem (and, later on, with $N = 6$ in the Faddeev eqs.).

Now for $r_0(z) = (h_0 - z)^{-1}$ set

$$\big(r_0(z)f_1, f_2\big) \equiv \int_{\mathbb{R}^3} d\boldsymbol{q} \frac{f_1(\boldsymbol{q}) f_2(\boldsymbol{q})}{\boldsymbol{q}^2 - z} \qquad \big(= \Phi(z), \quad \boldsymbol{\lambda} = 0\big),$$

where f_1 and f_2 are holomorphic function. Then by Lemma 1

$$\big(r_0(z)f_1, f_2\big)|_{\Pi_1} = \big(r_0(z)f_1, f_2\big)|_{\Pi_0} - \boldsymbol{\pi}\mathrm{i}\sqrt{z} \int_{S^2} d\widehat{\boldsymbol{q}} f_1(\sqrt{z}\widehat{\boldsymbol{q}}) f_1(\sqrt{z}\widehat{\boldsymbol{q}})$$

$$\Updownarrow$$

$$r_0(z)|_{\Pi_1} = r_0(z) + \mathrm{a}_0(z)\mathrm{j}^\dagger(z)\mathrm{j}(z),$$

where $\mathrm{a}_0(z) = -\boldsymbol{\pi}\mathrm{i}\sqrt{z}$, and

$$\big(\mathrm{j}(z)f\big)(\widehat{\boldsymbol{k}}) = f(\sqrt{z}\widehat{\boldsymbol{k}}).$$

What can be said about the operator $t(z)$?

$$t(z) = V - V r_0 t.$$

After the continuation to the sheet Π_1 we obtain

$$t' = V - V(r_0 + \mathrm{a}_0 \mathrm{j}^\dagger \mathrm{j}) t', \qquad t' = t|_{\Pi_1},$$

which implies that

$$(I + V r_0) t' = V - \mathrm{a}_0 V \mathrm{j}^\dagger \mathrm{j} t'.$$

Perform inversion of $(I + V r_0)$ taking into account that $t(z) = V - V r_0 t$ and, hence, $(I + V r_0)^{-1} V = t$:

$$t' = t - \mathrm{a}_0 \, t \mathrm{j}^\dagger \mathrm{j} \, t'. \tag{3.3}$$

Further on, apply j to the both parts and get

$$\mathrm{j}t' = \mathrm{j}\,t - \mathrm{a}_0 \mathrm{j}t\mathrm{j}^\dagger\,\mathrm{j}t',$$

which means

$$(I + \mathrm{a}_0 \mathrm{j}t\mathrm{j}^\dagger)\mathrm{j}t' = \mathrm{j}t. \tag{3.4}$$

Notice that

$$\boxed{I + \mathrm{a}_0 \mathrm{j}t\mathrm{j}^\dagger = s(z)} \qquad \text{is the scattering matrix,}$$

$$s(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{k}}', z) = \delta(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{k}}') - \pi\mathrm{i}\sqrt{z}\, t(\sqrt{z}\widehat{\boldsymbol{k}}, \sqrt{z}\widehat{\boldsymbol{k}}', z).$$

Hence,

$$\mathrm{j}t' = [s(z)]^{-1}\mathrm{j}t$$

Come back to Eq. (3.3) and conclude that

$$\boxed{t' = t - \mathrm{a}_0\, t\mathrm{j}^\dagger\, [s(z)]^{-1}\, \mathrm{j}t,} \tag{3.5}$$

$$\mathrm{a}_0(z) = -\pi \mathrm{i}\sqrt{z},$$

that is

$$\boxed{t(z)|_{\Pi_1} = t(z) - \mathrm{a}_0(z)\, t(z)\mathrm{j}^\dagger(z)\, [s(z)]^{-1}\, \mathrm{j}(z)\, t(z).} \tag{3.6}$$

All the entries on the r.h.s. of (3.6) ***are taken from the physical sheet!***

From (3.6) we derive that

$$\boxed{s(z)|_{\Pi_1} = \mathscr{E}\,[s(z)]^{-1}\,\mathscr{E},}$$

where $\mathscr{E}$ is the inversion, $(\mathscr{E} f)(\widehat{\boldsymbol{k}}) = f(-\widehat{\boldsymbol{k}})$.

In a similar way,

$$\boxed{r(z)|_{\Pi_1} = r + \mathrm{a}_0\,(I - rV)\mathrm{j}^{\dagger}\,[s(z)]^{-1}\mathrm{j}(I - Vr).}$$

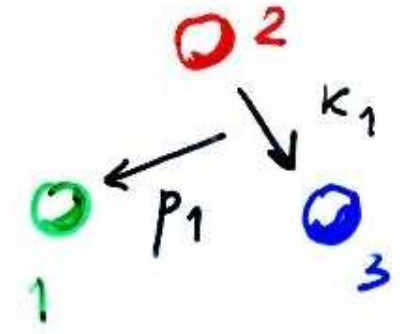
Hence the resonances are nothing but zeros of $s(z)$ in the physical sheet, that is,

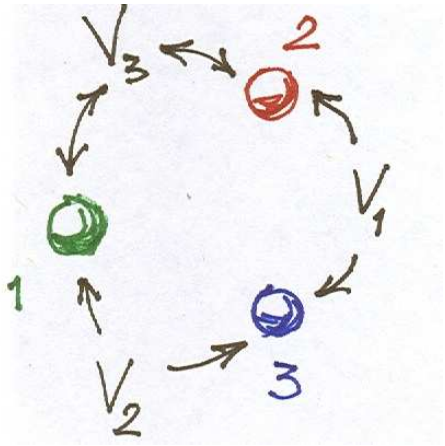
$$\boxed{z \text{ is a resonance} \iff \text{there is } \mathscr{A} \text{ on } S^2 \text{ such that } s(z)\mathscr{A} = 0.}$$

$\mathscr{A}$ is the breakup amplitude of the resonance state, i.e. the corresponding "Gamov vector" (the resonance solution to the Schrödinger equation) has the following asymptotics

$$\psi_{\text{res}}(\boldsymbol{x}) \underset{\boldsymbol{x}\to\infty}{\sim} \mathscr{A}(-\widehat{\boldsymbol{x}})\frac{\mathrm{e}^{-\mathrm{i}\sqrt{z}|\boldsymbol{x}|}}{|\boldsymbol{x}|}.$$

4 Three-Body Problem





$$\boldsymbol{k}_\alpha = \left(\frac{\mathrm{m}_\beta + \mathrm{m}_\gamma}{2\mathrm{m}_\beta \mathrm{m}_\gamma}\right)^{1/2} \frac{\mathrm{m}_\beta \mathbf{p}_\gamma - \mathrm{m}_\gamma \mathbf{p}_\beta}{\mathrm{m}_\beta + \mathrm{m}_\gamma},$$

$$\boldsymbol{p}_\alpha = \left(\frac{\mathrm{m}_1 + \mathrm{m}_2 + \mathrm{m}_3}{2\mathrm{m}_\alpha(\mathrm{m}_\beta + \mathrm{m}_\gamma)}\right)^{1/2} \frac{(\mathrm{m}_\beta + \mathrm{m}_\gamma)\mathbf{p}_\alpha - \mathrm{m}_\alpha(\mathbf{p}_\beta + \mathbf{p}_\gamma)}{\mathrm{m}_1 + \mathrm{m}_2 + \mathrm{m}_3}$$

Here, (α, β, γ) is a cyclic permutation of the indices $(1,2,3)$.

$$H_0 = \boldsymbol{k}_\alpha^2 + \boldsymbol{p}_\alpha^2, \qquad V = V_1 + V_2 + V_3, \qquad H = H_0 + V$$

$$R_0(z) = (H_0 - z)^{-1}, \qquad R(z) = (H - z)^{-1}$$

T-operator: $\qquad T(z) = V - VR(z)V$

Faddeev components:

$$M_{\alpha\beta} = \delta_{\alpha\beta} V_\alpha - V_\alpha R(z) V_\beta \qquad (\alpha, \beta = 1,2,3)$$

Faddeev equations in the matrix form:

$$\boxed{M(z) = \mathbf{t}(z) - \mathbf{t}(z)\mathbf{R}_0(z)\Upsilon M(z)\,.}$$

Here

$$\mathbf{R}_0 = \begin{pmatrix} R_0 & 0 & 0 \\ 0 & R_0 & 0 \\ 0 & 0 & R_0 \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} \mathbf{t}_1 & 0 & 0 \\ 0 & \mathbf{t}_2 & 0 \\ 0 & 0 & \mathbf{t}_3 \end{pmatrix}$$

with

$$\boxed{\mathbf{t}_\alpha(P, P', z) = t_\alpha(\boldsymbol{k}_\alpha, \boldsymbol{k}'_\alpha, z - \boldsymbol{p}_\alpha^2)\,\delta(\boldsymbol{p}_\alpha - \boldsymbol{p}'_\alpha).}$$

$$\Upsilon = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

Let h_α be the Hamiltonian of the two-body subsystem α, and ε_α and ψ_α the (only) binding energy and b.s. wave function, respectively, that is,

$$h_\alpha \psi_\alpha = \varepsilon_\alpha \psi_\alpha .$$

Then

$$t_\alpha(\boldsymbol{k}, \boldsymbol{k}', z) = -\frac{\varphi_\alpha(\boldsymbol{k})\overline{\varphi_\alpha(\boldsymbol{k}')}}{\varepsilon_\alpha - z} + \widetilde{t}_\alpha(\boldsymbol{k}, \boldsymbol{k}', z)$$

with the formfactor

$$\varphi_\alpha = V_\alpha \psi_\alpha .$$

Recall that

$$R_0(P, P', z) = \frac{\delta(P - P')}{P^2 - z}.$$

These kernels (associated with the corresponding thresholds) are the sources of the Cauchy type integrals in Faddeev equations.

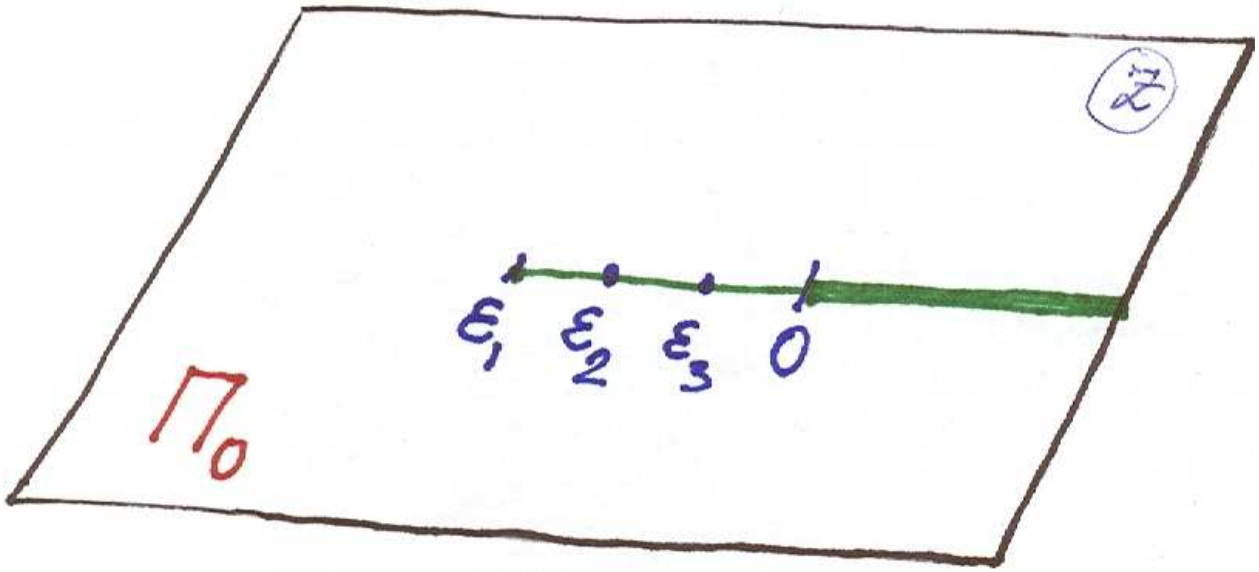
Further, we perform the analytic continuation of the Faddeev equations. It is remarkable that

the continued Faddeev equations can be solved explicitly (!)
— in terms of the matrix M itself,

and the "values" of $M(z)$ are taken exclusively from the physical energy sheet. The situation is very the same as in the case of the two-body T-matrix.

Surely, the result of continuation depends on the unphysical sheet under consideration.

How many sheets do we have in the three-body case?



Two-body binding energies ε_1, ε_2, ε_3 are square root branching points
The three-body threshold 0 is a logarithmic branching point

Hence, only encircling the two-body thresholds one arrives at **seven** different unphysical sheets.

The three-body threshold generates infinitely many unphysical sheets.

There is also a "fine structure": in particular, additional branching points, already on the unphysical sheets, may be generated by the two-body resonances. We did not yet have a look at the unphysical sheets of the "second order".

In order to enumerate the sheets (of the first order only) we need a multi-index,

$$l = (l_0, l_1, l_2, l_3),$$

with

$$l_0 = \ldots, -1, 0, 1, \ldots \qquad (0 \text{ physical}, \pm 1, \pm 2, \ldots \text{ unphysical})$$
$$l_\alpha = 0, 1 \qquad (0 \text{ physical}, 1 \text{ unphysical})$$

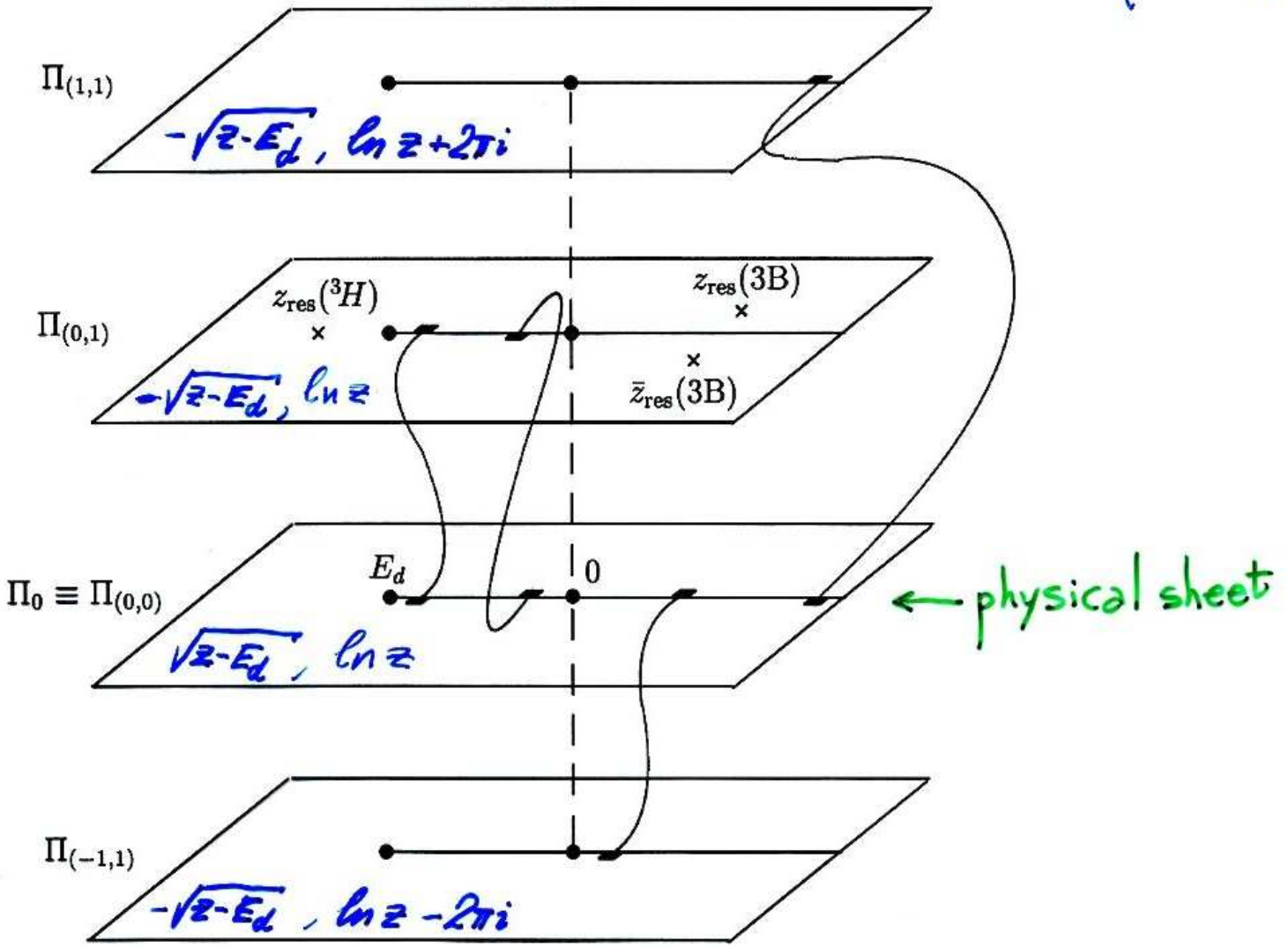
Π_l the corresponding unphysical sheet

Also introduce

$$L = \begin{pmatrix} l_0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 \\ 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & l_3 \end{pmatrix} \qquad \text{and} \qquad \widetilde{L} = \begin{pmatrix} |l_0| & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 \\ 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & l_3 \end{pmatrix}.$$

nnp, 3B, $^4He_3$ trimer

$$f(z)=\begin{pmatrix}\ln z\\ (z-E_d)^{1/2}\end{pmatrix}$$



$\Pi_{(l_0,l_1)}$

Only physical sheet and **three neighboring** sheets of **infinitely many** unphysical sheets are shown here: the only two-cluster unphysical sheet and two three-body unphysical sheets.

In the simple four-channel case under consideration the three-body scattering matrix is a 4×4 operator matrix of the form

$$S(z)=\widehat{I}+A(z)\widehat{\mathscr{T}}(z),$$

$$\widehat{\mathscr{T}}=\begin{pmatrix}\widehat{\mathscr{T}}_{00} & \widehat{\mathscr{T}}_{01} & \widehat{\mathscr{T}}_{02} & \widehat{\mathscr{T}}_{03}\\ \widehat{\mathscr{T}}_{10} & \widehat{\mathscr{T}}_{11} & \widehat{\mathscr{T}}_{12} & \widehat{\mathscr{T}}_{13}\\ \widehat{\mathscr{T}}_{20} & \widehat{\mathscr{T}}_{21} & \widehat{\mathscr{T}}_{22} & \widehat{\mathscr{T}}_{23}\\ \widehat{\mathscr{T}}_{30} & \widehat{\mathscr{T}}_{31} & \widehat{\mathscr{T}}_{32} & \widehat{\mathscr{T}}_{33}\end{pmatrix},$$

where

$$A(z)=\operatorname{diag}\{-\pi\mathrm{i}z^2,-\pi\mathrm{i}\sqrt{z-\varepsilon_1},-\pi\mathrm{i}\sqrt{z-\varepsilon_2},-\pi\mathrm{i}\sqrt{z-\varepsilon_3}\}$$

Up to a scalar function of z the kernel of the entry $\widehat{\mathscr{T}}_{\alpha\beta}$ coincides with the amplitude for the corresponding process,

$$\begin{aligned}
&\widehat{\mathscr{T}}_{00}: && 3\longrightarrow 3\\
&\widehat{\mathscr{T}}_{\alpha 0}: && 2\longrightarrow 3, \quad \alpha=1,2,3\\
&\widehat{\mathscr{T}}_{0\beta}: && 3\longrightarrow 2, \quad \beta=1,2,3\\
&\widehat{\mathscr{T}}_{\alpha\beta}: && 2\longrightarrow 2, \quad \alpha,\beta=1,2,3
\end{aligned}$$

The matrix $\widehat{\mathscr{T}}$ is obtained from the matrix

$$\mathscr{T} = \begin{pmatrix} \mathscr{T}_{00} & \mathscr{T}_{01} & \mathscr{T}_{02} & \mathscr{T}_{03} \\ \mathscr{T}_{10} & \mathscr{T}_{11} & \mathscr{T}_{12} & \mathscr{T}_{13} \\ \mathscr{T}_{20} & \mathscr{T}_{21} & \mathscr{T}_{22} & \mathscr{T}_{23} \\ \mathscr{T}_{30} & \mathscr{T}_{31} & \mathscr{T}_{32} & \mathscr{T}_{33} \end{pmatrix}$$

with elements

$$\begin{aligned}
\mathscr{T}_{00}(z) &= T(z) = V - VR(z)V, && \Big| \quad \mathscr{T}_{00}(P,P',z) \\
\mathscr{T}_{0\beta}(z) &= \big(\overline{V}_\beta - VR(z)\overline{V}_\beta\big)|\psi_\beta\rangle, && \Big| \quad \mathscr{T}_{0\beta}(P,p'_\beta,z) \\
\mathscr{T}_{\alpha 0}(z) &= \langle\psi_\alpha|\big(\overline{V}_\alpha - \overline{V}_\alpha R(z)V\big), && \Big| \quad \mathscr{T}_{\alpha 0}(p_\alpha,P',z) \\
\mathscr{T}_{\alpha\beta}(z) &= \langle\psi_\alpha|\big(\overline{V}_\alpha - \overline{V}_\alpha R(z)\overline{V}_\beta\big)|\psi_\beta\rangle, && \Big| \quad \mathscr{T}_{\alpha\beta}(p_\alpha,p'_\beta,z) \\
& \alpha,\beta = 1,2,3.
\end{aligned}$$

(!!) $U_{00} = T, \quad U_{0\beta} = \overline{V}_\beta - VR\overline{V}_\beta, \quad U_{\alpha 0} = \overline{V}_\alpha - \overline{V}_\alpha RV, \quad U_{\alpha\beta} = \overline{V}_\alpha - \overline{V}_\alpha R(z)\overline{V}_\beta$

— transition operators $\Longrightarrow$ **AGS equations:** refer to **Prof. Sandhas' lecture**

$$\begin{aligned}
\widehat{\mathscr{T}_{00}}(\widehat{P},\widehat{P}',z) &= \mathscr{T}_{00}(\sqrt{z}\widehat{P},\sqrt{z}\widehat{P}',z), \\
\widehat{\mathscr{T}_{0\beta}}(\widehat{P},\widehat{p}'_\beta,z) &= \mathscr{T}_{0\beta}(\sqrt{z}\widehat{P},\sqrt{z-\varepsilon_\beta}\widehat{p}'_\beta,z), \\
\widehat{\mathscr{T}_{\alpha 0}}(\widehat{p}_\alpha,\widehat{P}',z) &= \mathscr{T}_{\alpha 0}(\sqrt{z-\varepsilon_\alpha}\widehat{p}_\alpha,\sqrt{z}\widehat{P}',z), \\
\widehat{\mathscr{T}_{\alpha\beta}}(\widehat{p}_\alpha,\widehat{p}'_\beta,z) &= \mathscr{T}_{\alpha 0}(\sqrt{z-\varepsilon_\alpha}\widehat{p}_\alpha,\sqrt{z-\varepsilon_\beta}\widehat{p}'_\beta,z),
\end{aligned}$$

Explicitly solving the continued Faddeev equations results in the following

$$\boxed{M|_{\Pi_l} = M + Q_M L S_l^{-1} \widetilde{L} \widetilde{Q}_M.}$$

where Q_M and $\widetilde{Q}_M$ are explicitly written in terms of the Faddeev components $M_{\alpha\beta}$ taken immediately from the physical sheet. S_l is a "truncation" of the total three-body scattering matrix S,

$$S_l = I + A(z) L \widehat{\mathscr{T}} \widetilde{L}.$$

Similarly,

$$\boxed{R|_{\Pi_l} = R + Q_R L S_l^{-1} \widetilde{L} \widetilde{Q}_R.}$$

Therefore, the singularities of $M(z)|_{\Pi_l}$ and $S(z)|_{\Pi_l}$ (as well as the ones of $R(z)|_{\Pi_l}$)) are determined by the inverse truncated scattering matrix in $S_l(z)^{-1}$.

$\widetilde{L}$ is nothing but a projection! — An example at the blackboard.

Thus, to find the resonances on the sheet Π_l one should simply look for the zeros of the truncated scattering matrix $S_l(z)$ in the physical sheet, that is, for the points z where $S_l(z)$ has eigenvalue zero:

$$S_l(z)\mathscr{A} = 0.$$

The vector $\mathscr{A}$ will consist of breakup amplitudes of the resonance state into the channels 0, 1, 2, and 3,

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_0(\widehat{X}) \\ \mathscr{A}_1(\widehat{\boldsymbol{y}}_1) \\ \mathscr{A}_0(\widehat{\boldsymbol{y}}_2) \\ \mathscr{A}_0(\widehat{\boldsymbol{y}}_3) \end{pmatrix} \qquad \text{(in coordinate space).}$$

To this end one can employ any approach that allows to calculate the corresponding truncation of the scattering matrix (surely, only for the energies z in the physical sheet). That is, any approach that allows to calculate the appropriate scattering, rearrangement and breakup amplitudes.

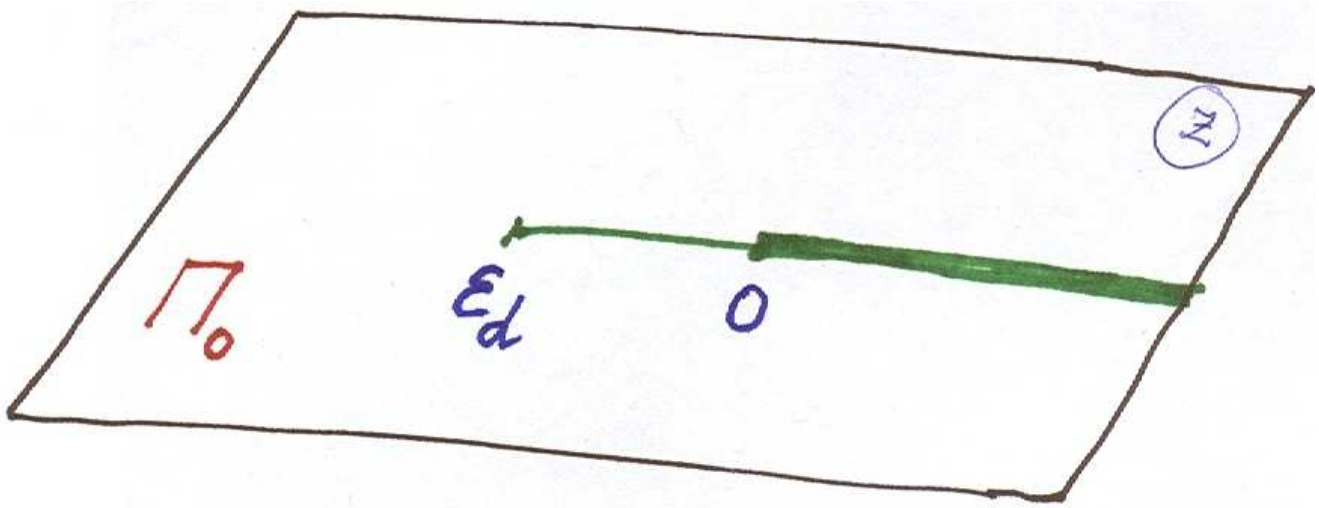
5 Configuration space. Applications

In order to find the amplitudes involved in S_l, one can use in particular the Faddeev differential equations.

We have employed the two-dimensional partial-wave Faddeev equations (arising as the result of a decomposition of the 6-dimensional Faddeev equations over bispherical harmonics).

- nnp system
- System of three bosons with nucleon masses
- ^{4}He three-atomic system

Refs.: E. A. Kolganova and AM, Phys. Atom. Nucl. **60** (1997), 177
AM and E. A. Kolganova, FBS Suppl. **10** (1999), 75
E. A. Kolganova and AM, Phys. Atom. Nucl. **62** (1999), 1179
E. A. Kolganova and AM, Comput. Phys. Comm. **126** (2000), 88
E. A. Kolganova, AM, and Y. K. Ho, Nucl. Phys. **A684** (2001), 623
E. A. Kolganova, AM, and Y. K. Ho, JCMSE **2** (2002), 149



With our computer code we could only calculate the $2 \to 2$ and $2 \to 3$ amplitudes. Hence we were restricted to the study of resonances on the two-cluster unphysical sheet, the one neighboring the physical sheet along the interval $(\varepsilon_d, 0)$.

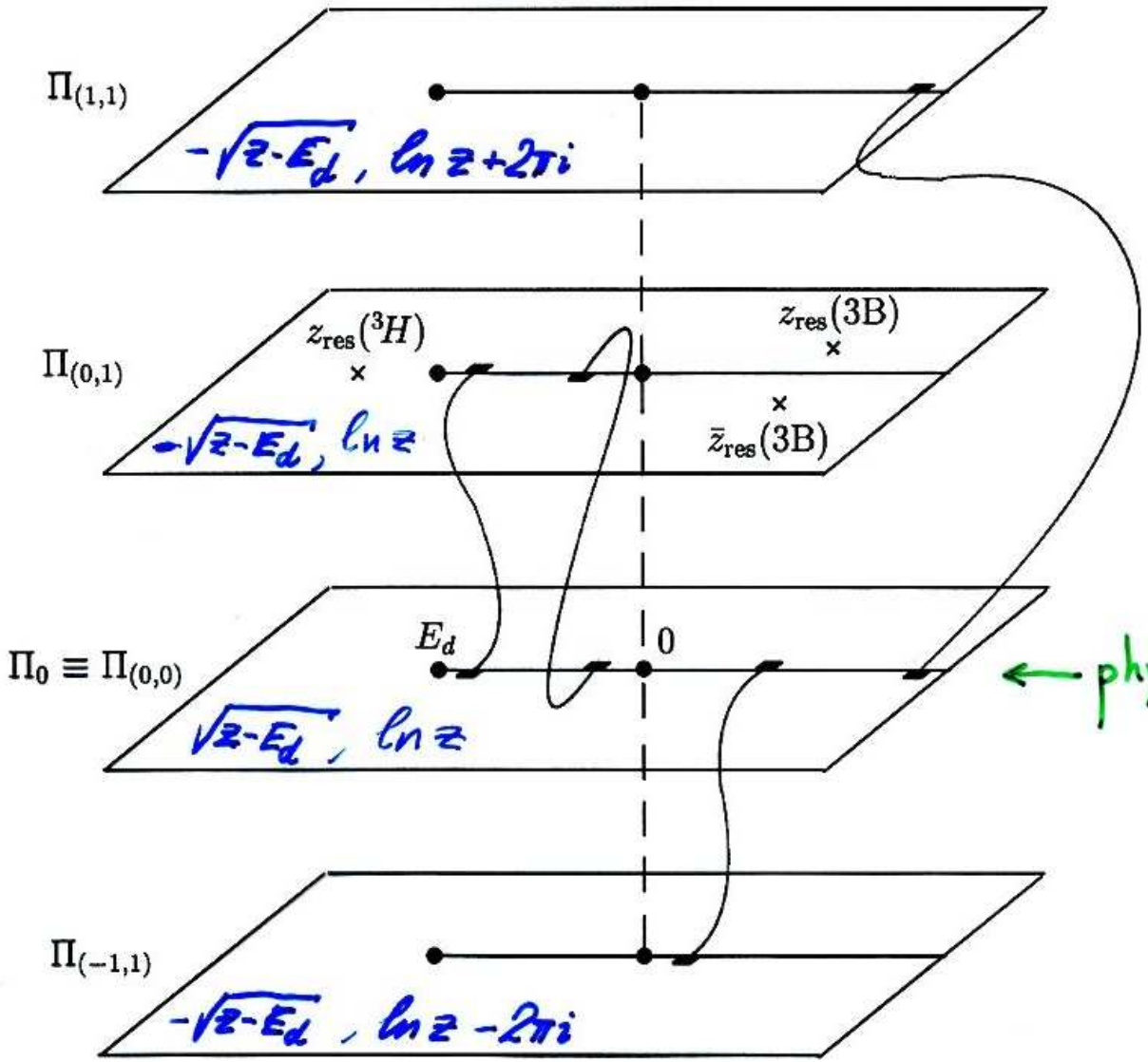
The resonances were looked for as zeros of the scattering matrix

$$S_{(0,1)}(z) = S_0(z) = 1 + 2\mathrm{i}a_0(z),$$

where $a_0(z)$ stands for the s-wave $2 \to 2$ elastic scattering amplitude.

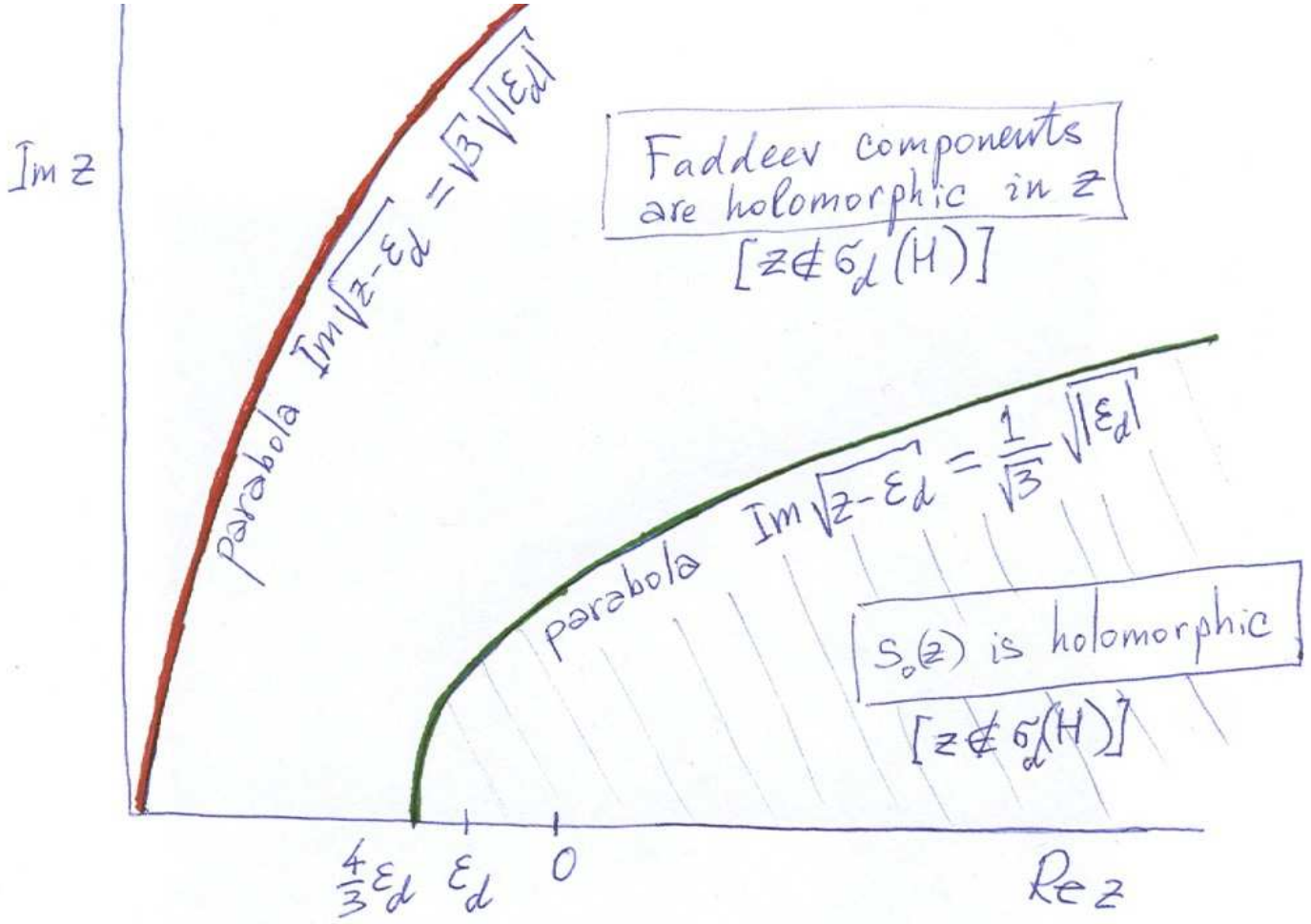
nnp, ^{3}B, ^{4}He$_3$ trimer

$$f(z) = \begin{pmatrix} \ln z \\ (z - E_d)^{1/2} \end{pmatrix}$$



← physical sheet

$\Pi_{(l_0, l_1)}$



Admissible domain in the case of three particles with the same mass; ε_d stands for the deuteron (or dimer) binding energy (the picture is borrowed from [*E. A. Kolganova and AM, Phys. Atom. Nucl.* **62** *(1999), 1179*]; for explanations see this paper).

An advantage: with this approach we can, of course, calculate virtual levels.

More details on formalism

Example of the symmetric $^4He_3$ system. Restrict to a total angular momentum $L = 0$. Two-dimensional integro-differential Faddeev equations

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+l(l+1)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)-E\right]\Phi_l(x,y)=\begin{cases}-V(x)\Psi_l(x,y), & x>c\\ 0, & x<c\,.\end{cases} \tag{5.1}$$

Here, x, y stand for the standard Jacobi variables and c for the core range. The angular momentum l corresponds to a dimer subsystem and a complementary atom; for an S-wave three-boson state. The partial wave function $\Psi_l(x,y)$ is related to the Faddeev components $\Phi_l(x,y)$ by

$$\Psi_l(x,y)=\Phi_l(x,y)+\sum_{l'}\int_{-1}^{+1}d\eta\, h_{ll'}(x,y,\eta)\,\Phi_{l'}(x',y'), \tag{5.2}$$

where

$$x'=\sqrt{\frac{1}{4}x^2+\frac{3}{4}y^2-\frac{\sqrt{3}}{2}xy\eta}\,,\qquad y'=\sqrt{\frac{3}{4}x^2+\frac{1}{4}y^2+\frac{\sqrt{3}}{2}xy\eta}\,,$$

and $1\le\eta\le 1$.

The functions $\Phi_l(x,y)$ satisfy the boundary conditions

$$\Phi_l(x,y)\,|_{x=0} = \Phi_l(x,y)\,|_{y=0} = 0\,. \tag{5.3}$$

Moreover, in the hard-core model they are required to satisfy the condition

$$\Phi_l(c,y) + \sum_{l'} \int_{-1}^{+1} d\eta\, h_{ll'}(c,y,\eta)\, \Phi_{l'}(x',y') = 0\,. \tag{5.4}$$

This guarantees the wave function $\Psi_l(x,y)$ to be zero not only at the core boundary $x = c$ but also inside the core domains.

The asymptotic boundary condition for the partial-wave Faddeev components of the two-fragment scattering states reads, as $\rho \to \infty$ and/or $y \to \infty$,

$$\begin{aligned} \Phi_l(x,y;p) \;=\; & \delta_{l0}\psi_d(x)\left\{\sin(py) + \exp(\mathrm{i}py)\left[a_0(p) + o\left(y^{-1/2}\right)\right]\right\} \\ & + \frac{\exp(\mathrm{i}\sqrt{E}\rho)}{\sqrt{\rho}}\left[A_l(\theta) + o\left(\rho^{-1/2}\right)\right]. \end{aligned} \tag{5.5}$$

Here, $\psi_d(x)$ is the dimer wave function, E stands for the scattering energy given by $E = \varepsilon_d + p^2$ with ε_d the dimer energy, and p for the relative momentum conjugate to the variable y. The variables $\rho = \sqrt{x^2+y^2}$ and $\theta = \arctan\dfrac{y}{x}$ are the hyperradius and hyperangle, respectively. The coefficient $a_0(p)$ is nothing but the elastic scattering amplitude, while the functions $A_l(\theta)$ provide us, at $E > 0$, with the corresponding partial-wave Faddeev breakup amplitudes.

$^4He_3$ [*E. A. Kolganova and AM, Phys. Atom. Nucl.* **62** *(1999), 1179*]

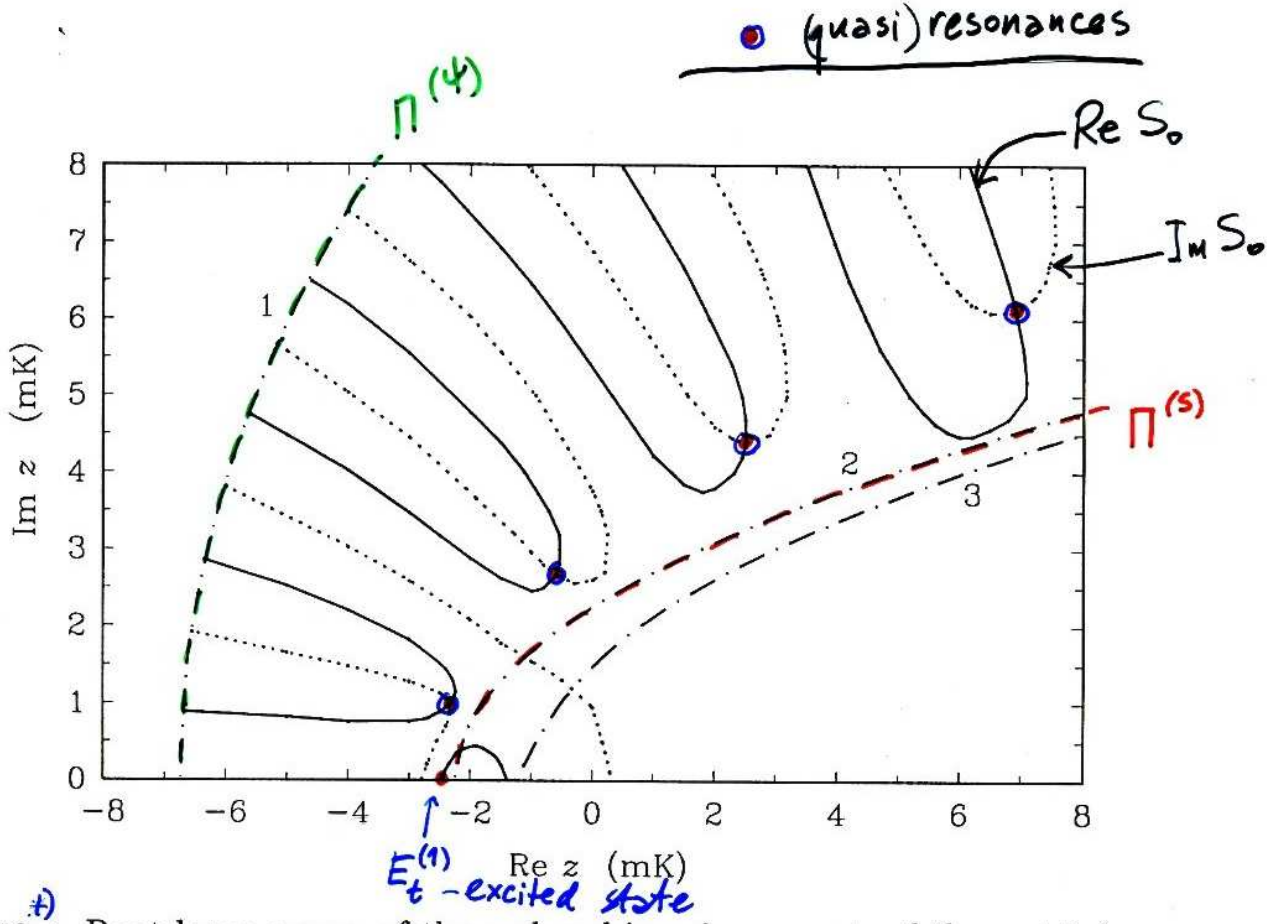


FIG. 1. Root locus curves of the real and imaginary parts of the scattering matrix $S_0(z)$. The solid lines correspond to $\mathrm{Re}\, S_0(z) = 0$ while the tiny dashed lines, to $\mathrm{Im}\, S_0(z) = 0$. The numbers 1, 2, 3 denote the boundaries of the domains $\Pi^{(\Psi)}$, $\Pi^{(S)}$ and $\Pi^{(A)}$, respectively. Complex roots of the function $S_0(z)$ are represented by the crossing points of the curves $\mathrm{Re}\, S_0(z) = 0$ and $\mathrm{Im}\, S_0(z) = 0$ and are located at $(-2.34 + \mathrm{i}\, 0.96)$ mK, $(-0.59 + \mathrm{i}\, 2.67)$ mK, $(2.51 + \mathrm{i}\, 4.34)$ mK and $(6.92 + \mathrm{i}\, 6.10)$ mK.

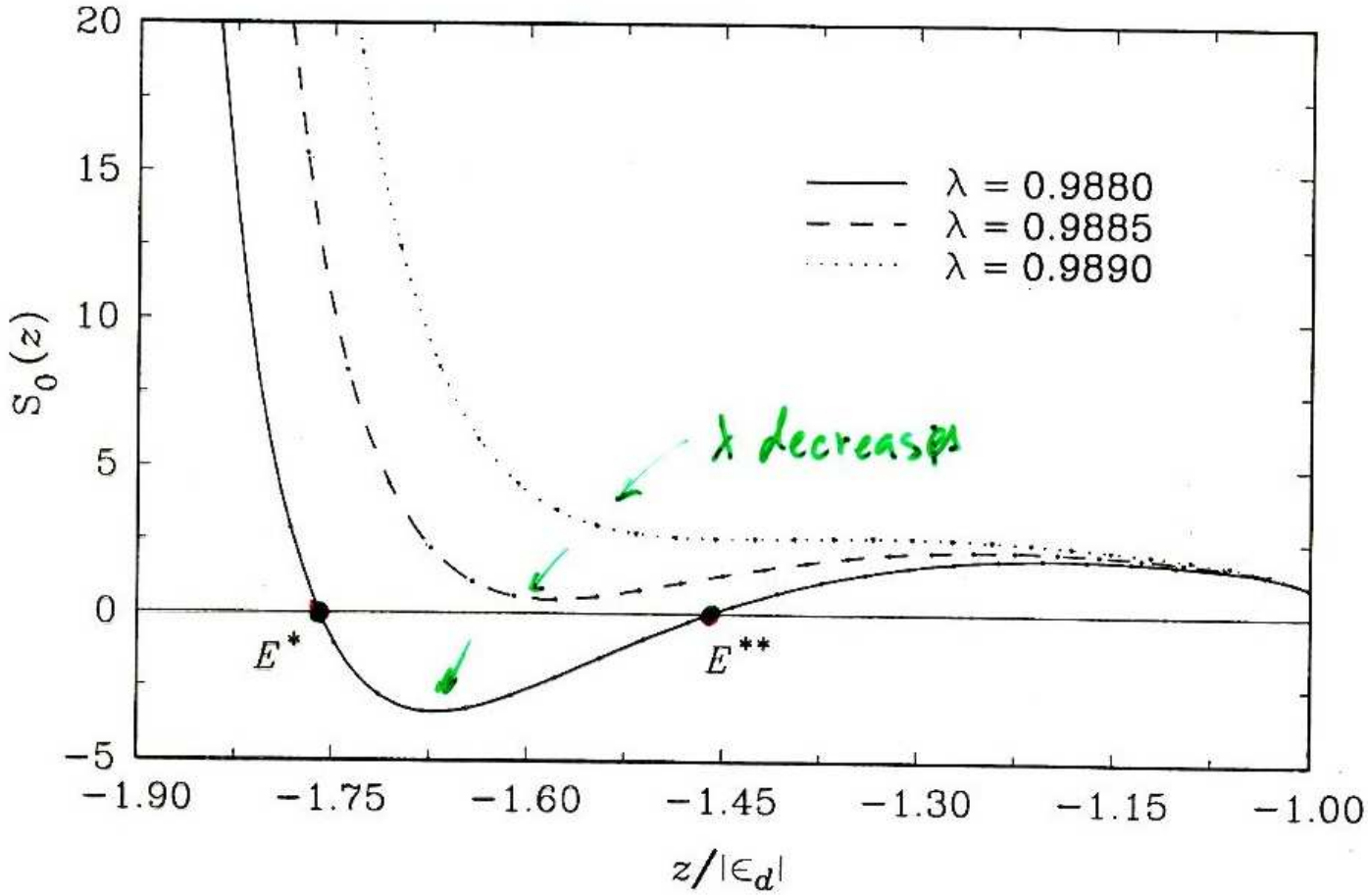
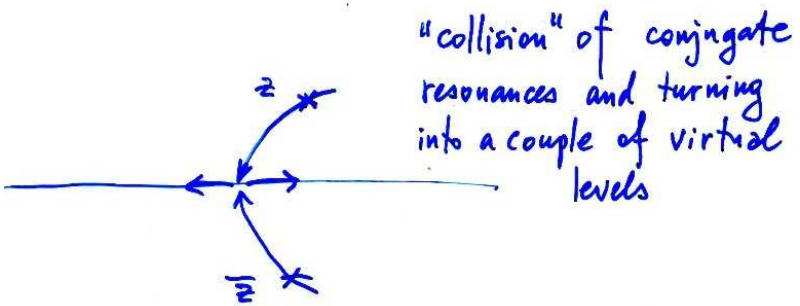


FIG. 2. Graphs of the function $S_0(z)$ at real $z \leq \epsilon_d$ for three values of $\lambda < 1$. The notations used: $E^* = E_t^{(2)*}/|\epsilon_d|$, $E^{**} = E_t^{(2)**}/|\epsilon_d|$.



The figure below has been borrowed from [E. Kolganova and AM, Proc. of 9th Intern. Conf. on Computational Modelling and Computing in Physics, p. 177]

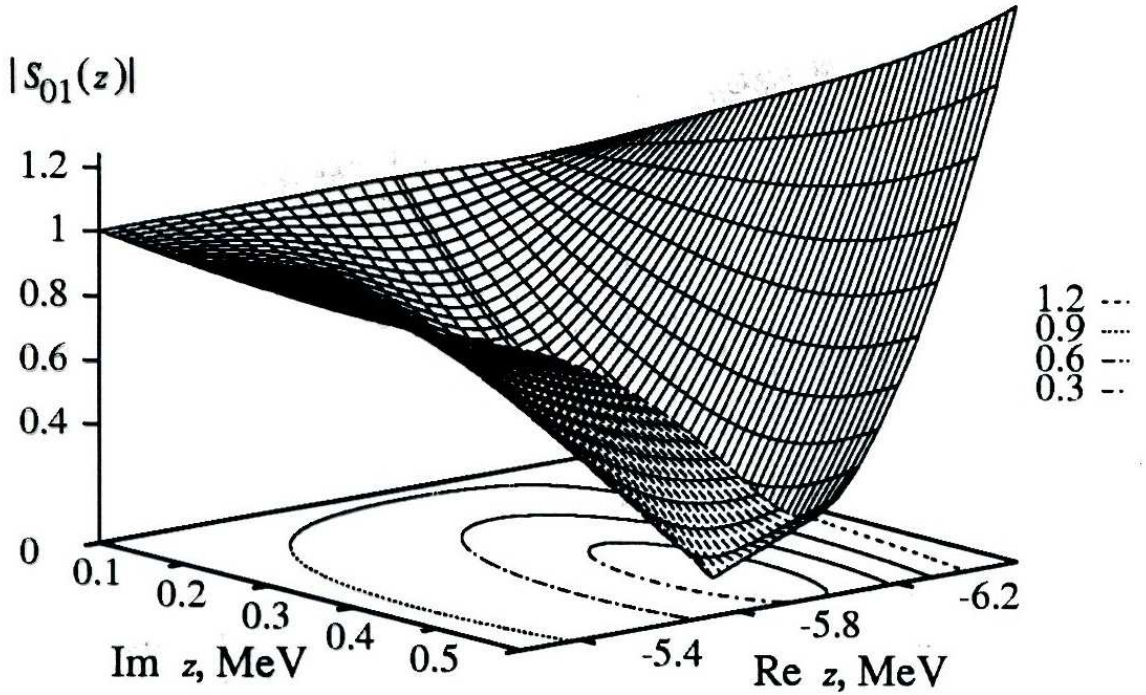


FIG. 2. Surface of the function $|S_{01}(z)|$ in the model system of three bosons with the nucleon masses. The potential $V^G(r)$ is used with the barrier $V_b = 1.5$ MeV. Position of the resonance $z_{\rm res}(3{\rm B})$ corresponds to the minimal (zero) value of $|S_{01}(z)|$.

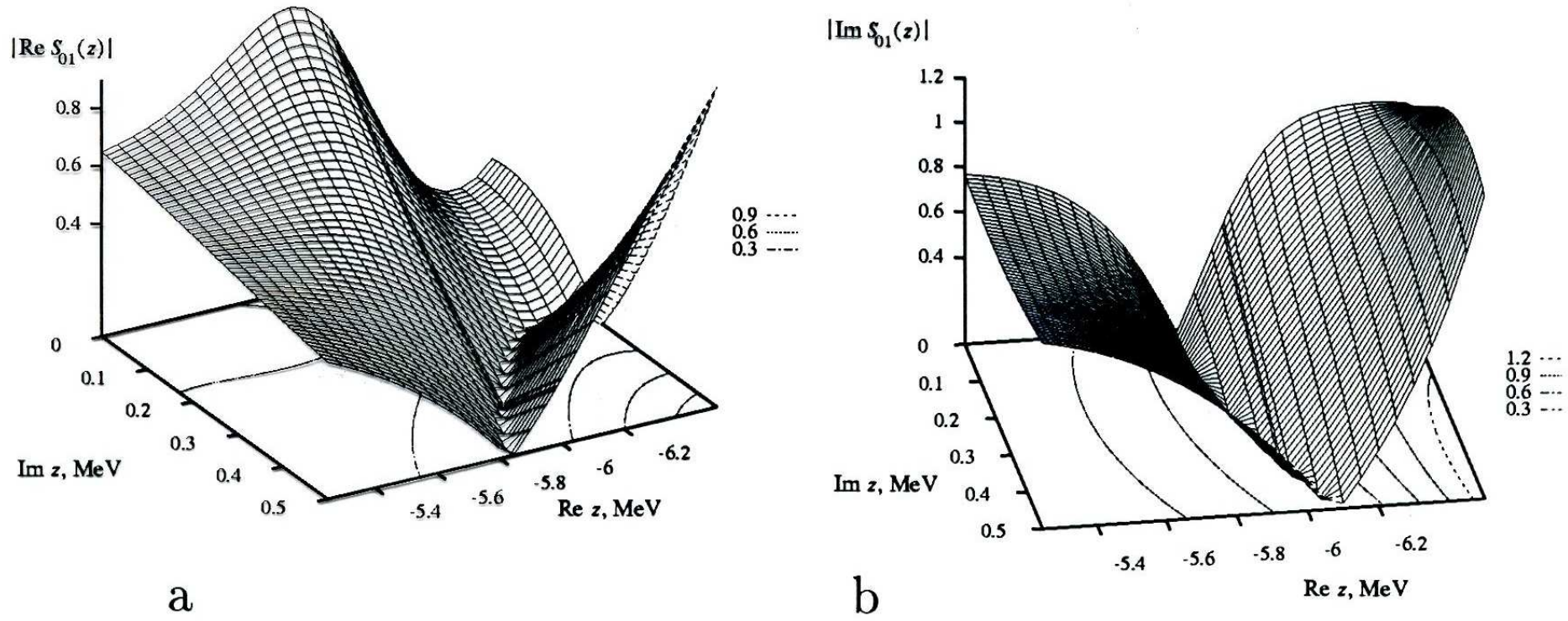


FIG. 3. Surface of the absolute values of real (a) and imaginary (b) components of the scattering matrix $S_{01}(z)$ in the model system of three bosons with the nucleon masses.

Conclusions

- Explicit representations for the three-body T-matrix, scattering matrix, and resolvent on unphysical energy sheets not only describe the structure of these quantities but also suggest the ways to calculate three-body resonances.
- A resonance on a sheet Π_l corresponds to a point z on the physical sheet where the truncated scattering matrix $S_l(z)$ has eigenvalue zero,

$$S_l(z)\mathscr{A} = 0.$$

- The corresponding eigenvector $\mathscr{A}$ consists of breakup amplitudes of the resonant state into various channels.
- We have also developed a numerical approach to calculation of three-body resonances based on the Faddeev differential equations in coordinate space. This approach has been successfully applied to several three-body systems (including nnp and $^4\mathrm{He}_3$). In particular the mechanism of emerging the Efimov states in the $^4\mathrm{He}_3$ has been studied.

6 Appendices

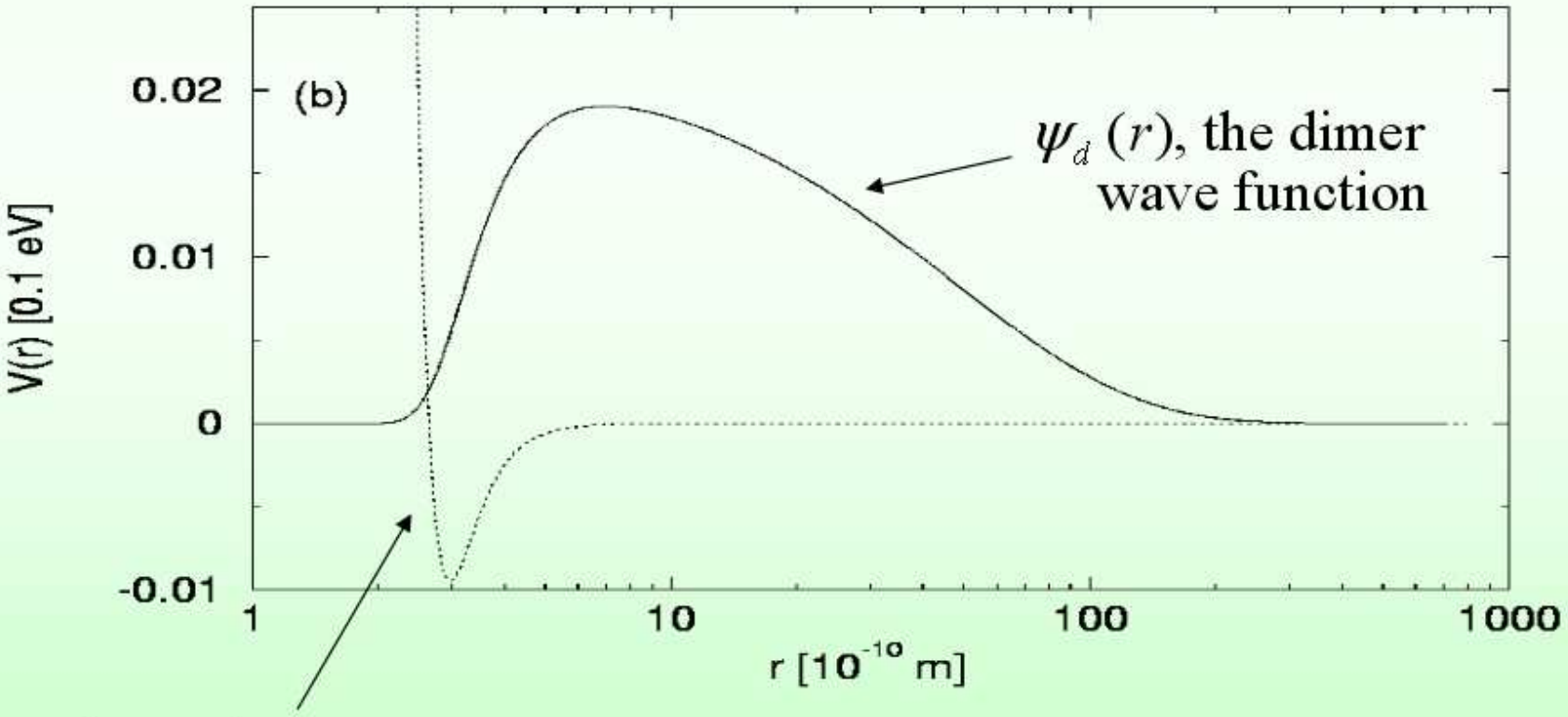
$^4He_2$

Two-body, theory

- Potential models: Aziz et al. – HFD-B (1987), LM2M2 (1991),Tang et al. – TTY (1995)

$$V_{\mathrm{HFD-B}}(x) = \varepsilon \left\{ A \exp(-\alpha\zeta + \beta\zeta^2) - \left[\frac{C_6}{\zeta^6} + \frac{C_8}{\zeta^8} + \frac{C_{10}}{\zeta^{10}} \right] F(\zeta) \right\}$$

where $\zeta = x/r_m$, and $F(\zeta) = \begin{cases} \exp\left[-(D/\zeta - 1)\right]^2, & \text{if } \zeta \le D \\ 1, & \text{if } \zeta > D\,. \end{cases}$



^{4}He - ^{4}He

Two-body, theory

- Potential models: Aziz et al. – HFD-B (1987), LM2M2 (1991)
  Tang et al. – TTY (1995)

TABLE I. Dimer energy ε_d, wave length $1/\varkappa^{(2)}$, and ^{4}He–^{4}He scattering length $\ell_{sc}^{(2)}$ for the potentials used, as compared to the experimental values of Ref. [5]. R. Grisenti, W. Schöllkopf, J. P. Toennies, G. C. Hegerfeld, T. Köhler, and M. Stoll, Phys. Rev. Lett. **85**, 2284 (2000).

	ε_d (mK)	$\ell_{sc}^{(2)}$(Å)	Potential	ε_d (mK)	$1/\varkappa^{(2)}$ (Å)	$\ell_{sc}^{(2)}$ (Å)
			LM2M2	−1.30348	96.43	100.23
Expt.	$1.1^{+0.3}_{-0.2}$	104^{+8}_{-18}	TTY	−1.30962	96.20	100.01
			HFD-B	−1.68541	84.80	88.50

Three-body, theory

bound state

$^4He_3$ Bound state calculations

Carbonell,Gignoux, Merkuriev (1993)

Cornelius, Gloeckle (1986)

Roudnev, Yakovlev (1999)

Panharipande et al, (1983)

Barnett, Walley (1993)

Lewerenz (1997)

Nielsen,Fedorov,Jensen (1998)

Esry,Lin,Greene (1996)

TABLE II. Ground state energy results, $|E_t^{(0)}|$ (in K).

		Faddeev equations				Variational methods			Adiabatic approaches	
Potential	l_{max}	[*]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
HFDHE2	0	0.0823		0.082					0.098	
	2	0.1124	0.107	0.11						
	4	0.1167			0.1171	0.1173				
HFD-B	0	0.0942	0.096							
	2	0.1277	0.130							
	4	0.1325			0.1330		0.133	0.131		
LM2M2	0	0.0891							0.106	
	2	0.1213								
	4	0.1259			0.1264					0.1252
TTY	0	0.0890								
	2	0.1212								
	4	0.1258			0.1264			0.126		

Motovilov,Kolganova, Sandhas,Sofianos (1997,2001)

$^4He_3$ Bound state calculations

TABLE III: Excited state energy results, $|E_t^{(1)}|$ (in mK).

Cornelius, Gloeckle (1986) · Nakaichi-Maeda, Lim (1983) · Roudnev, Yakovlev (2000) · Esry,Lin,Greene (1996) · Nielsen,Fedorov,Jensen (1998)

Potential	l_{max}	[*]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
HFDHE2	0	1.46	1.46	1.04		1.517			
	2	1.65	1.6						
	4	1.67			1.665				
HFD-B	0	2.45							
	2	2.71							
	4	2.74			2.734				2.735
LM2M2	0	2.02				2.118			
	2	2.25							
	4	2.28			2.271		2.269	2.19	2.265
TTY	0	2.02							
	2	2.25							
	4	2.28			2.280				2.277

Blume,Greene (2000)

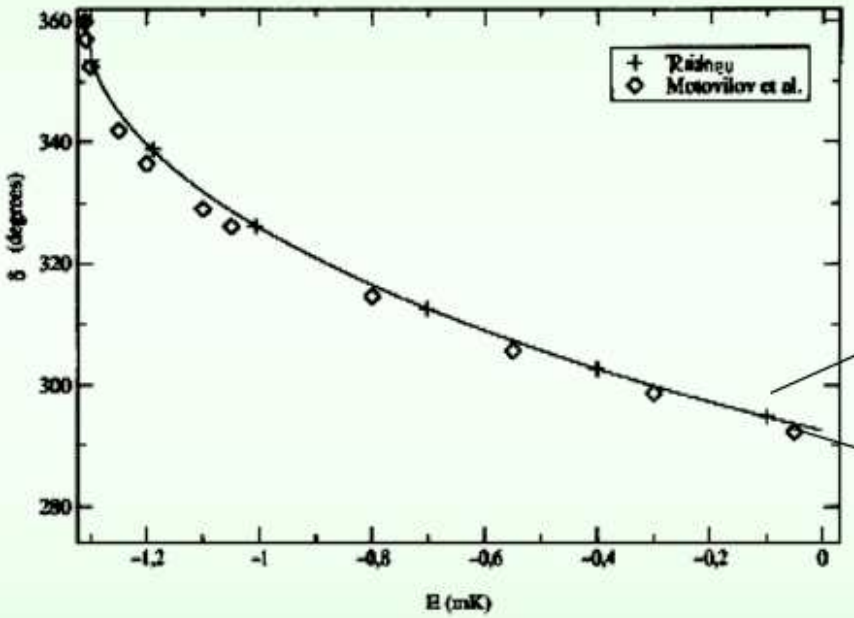
Barletta, Kievsky (2001)

Motovilov,Kolganova, Sandhas,Sofianos (1997,2001)

^4He - $^4He_2$

Three-body, theory scattering

Phase shifts calculations using Faddeev differential equations

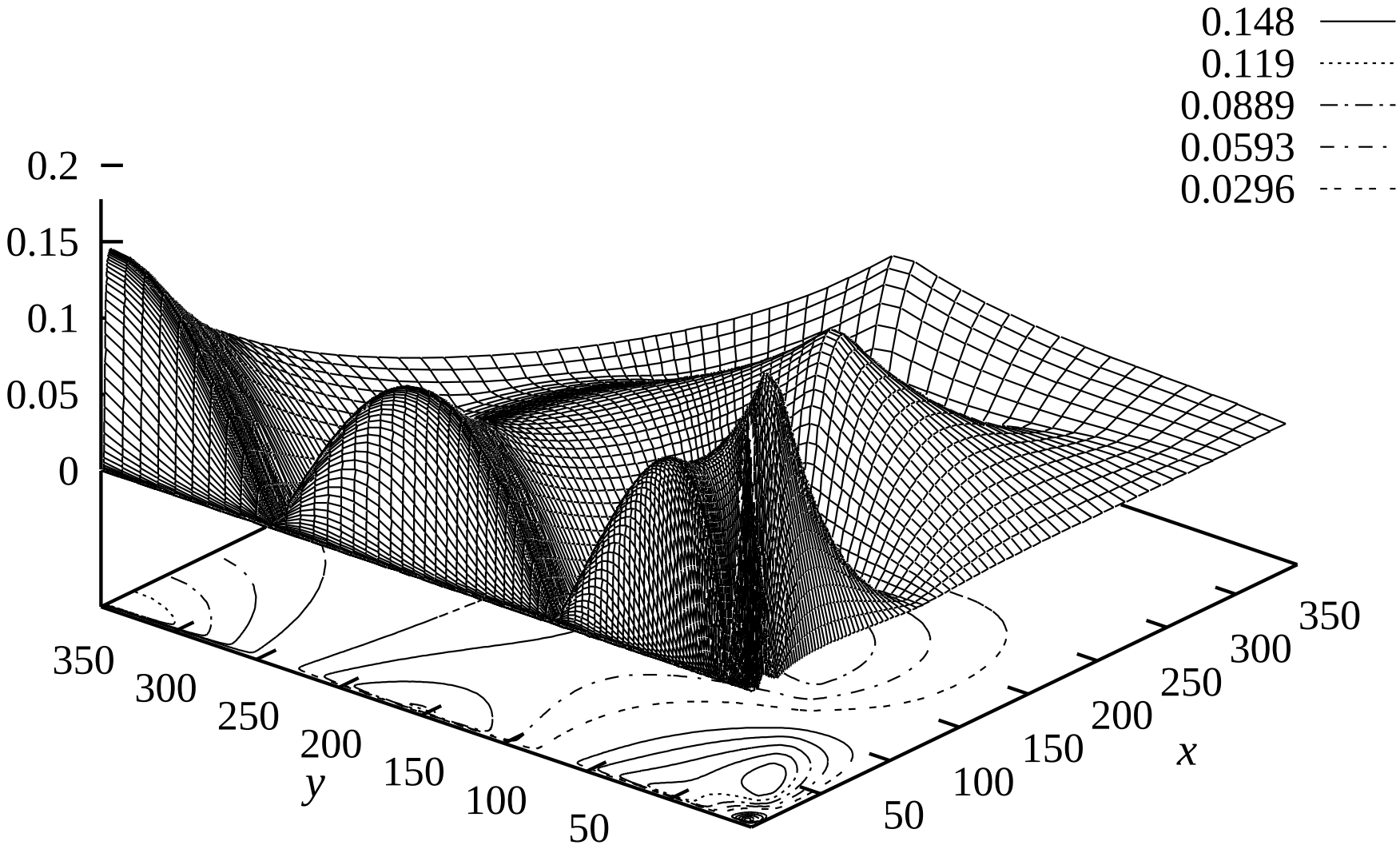


Roudnev (2003)

Kolganova,Motovilov(1998,2001)

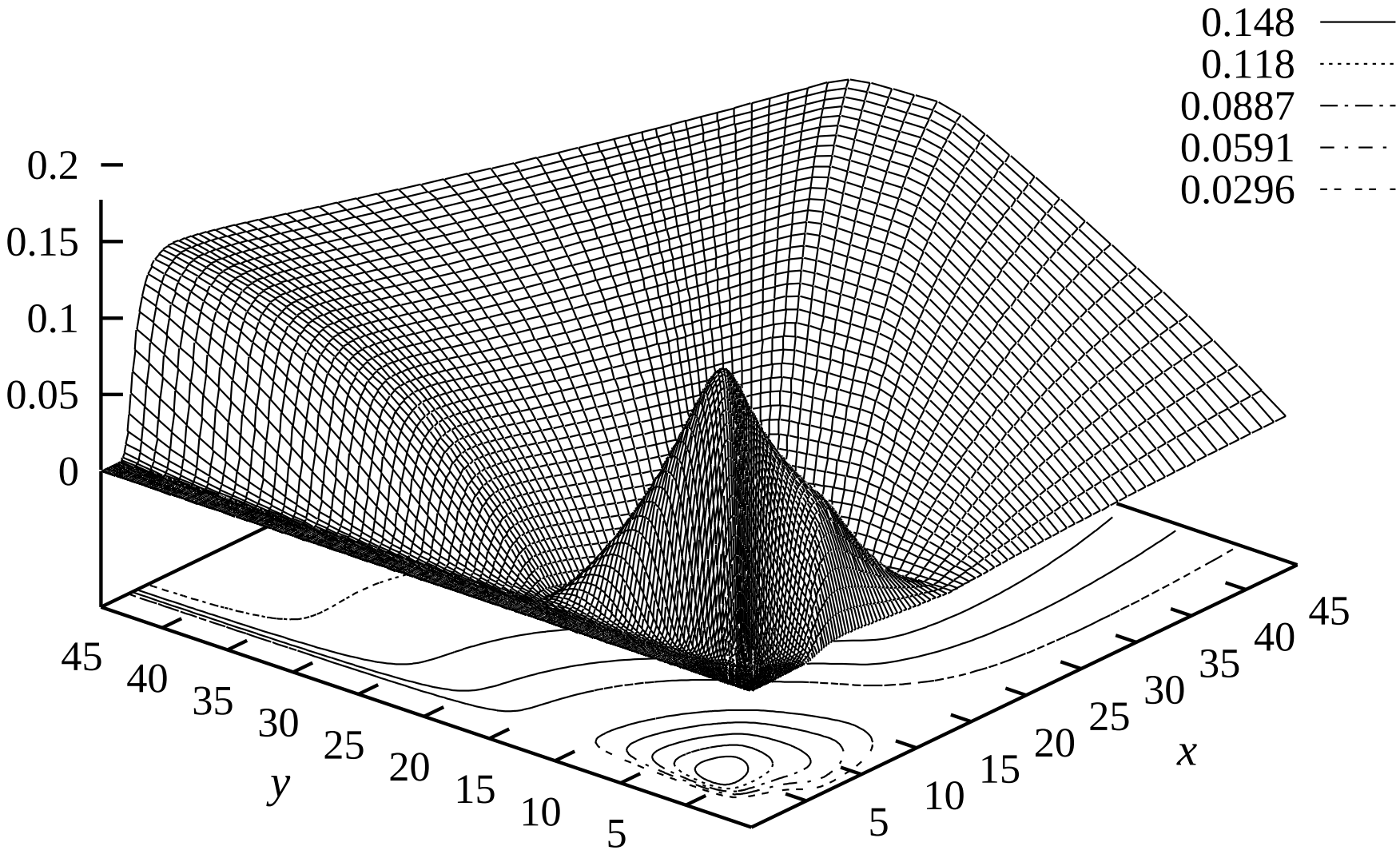
Fig. 2. He–He_2 scattering phase shifts for TTY potential. The energy *E* is the energy of the system in the center of mass frame.

Zero-range model - Hammer *et al.* (1999,2003), Penkov(2003)

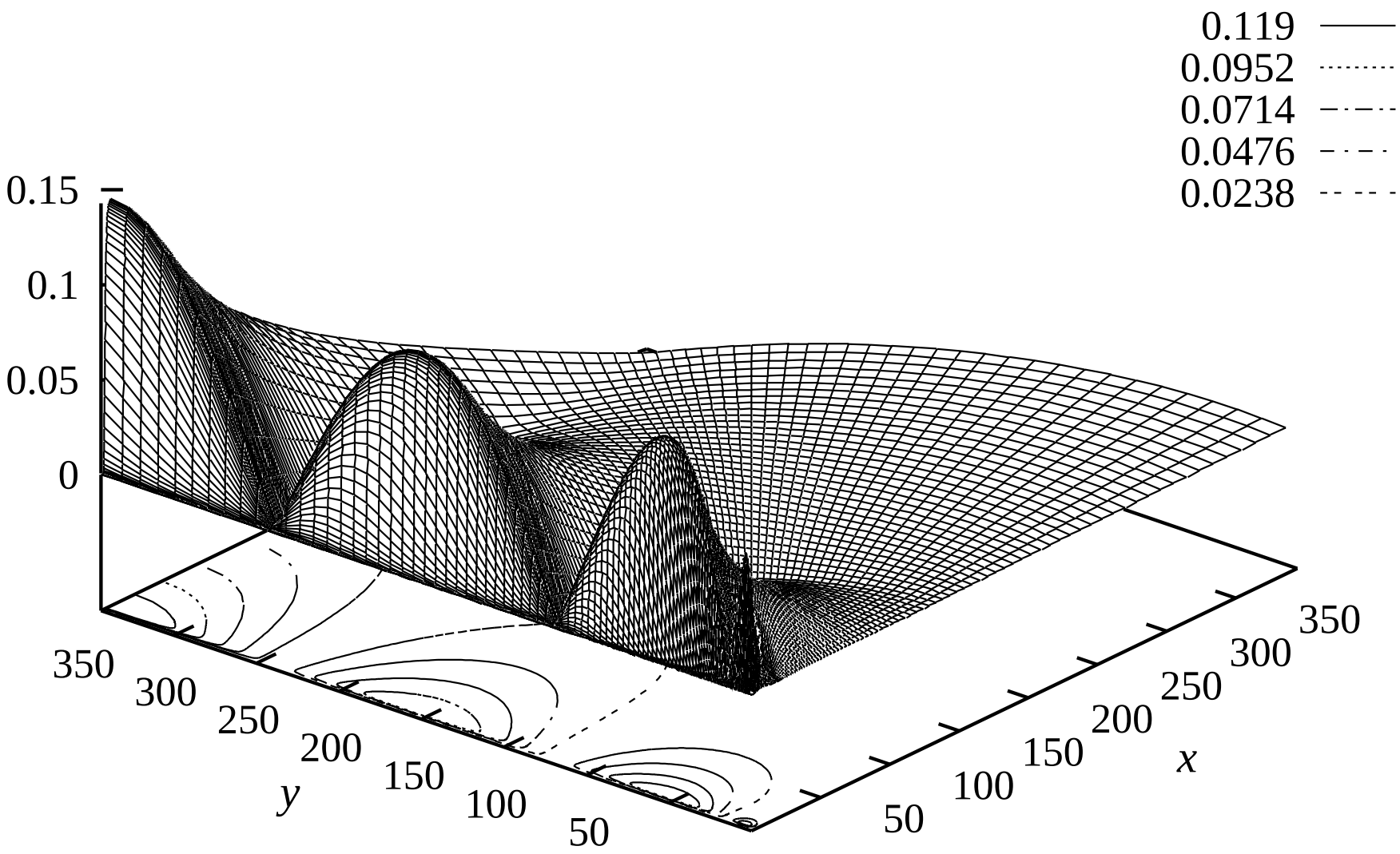


Absolute value of the ^{4}He–^{4}He$_2$ scattering wave function component $\Psi_0(x, y, p)$ for the HFD-B ^{4}He–^{4}He potential at $E = +1.4\,\mathrm{mK}$.

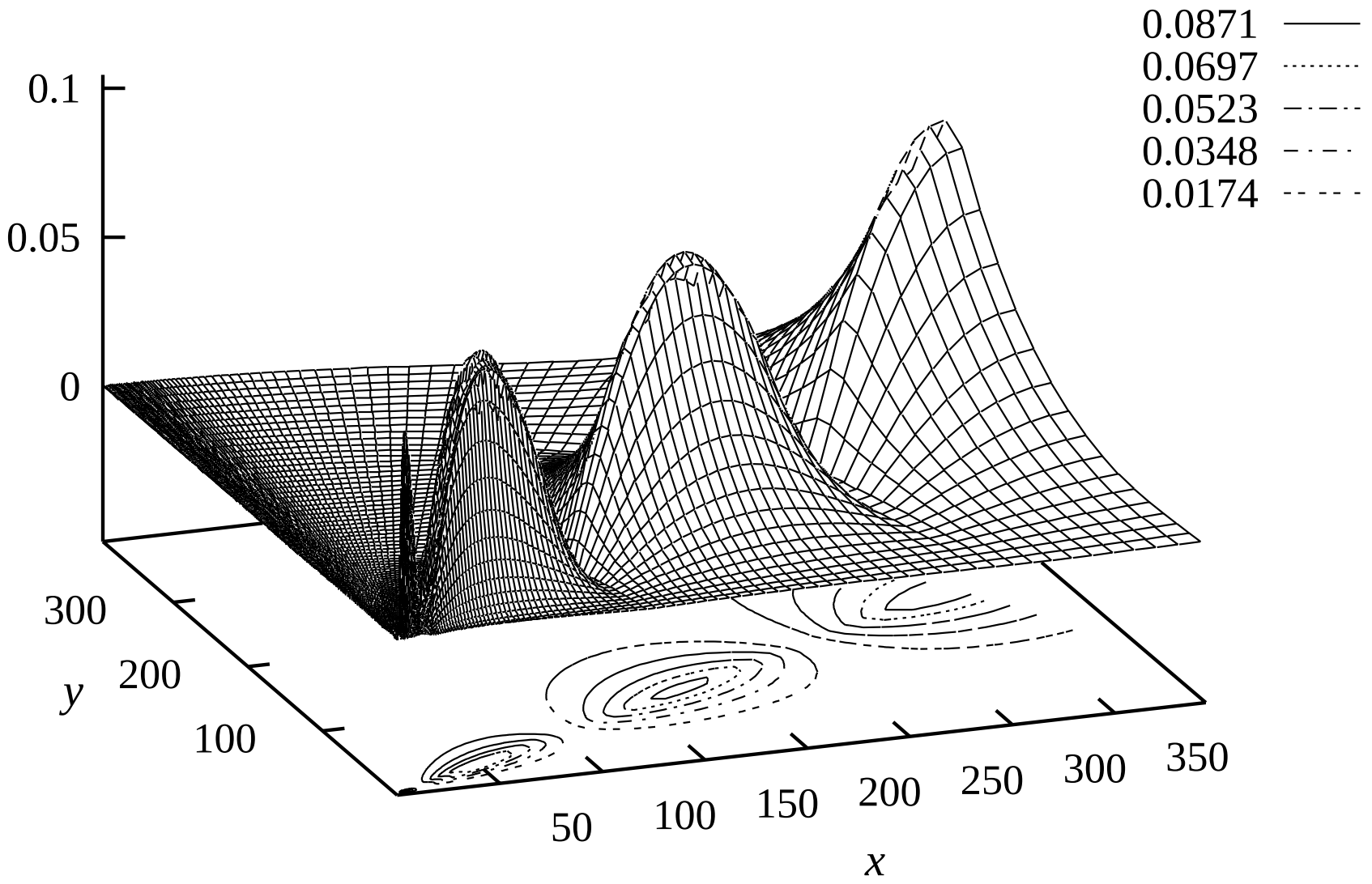
This picture and five pictures below are borrowed from [E. A. Kolganova, AM, and S. A. Sofianos, J. Phys. B **31** (1998), 1279] and/or [AM, S. A. Sofianos, and E. A. Kolganova, Chem. Phys. Lett. **275** (1997), 168] .



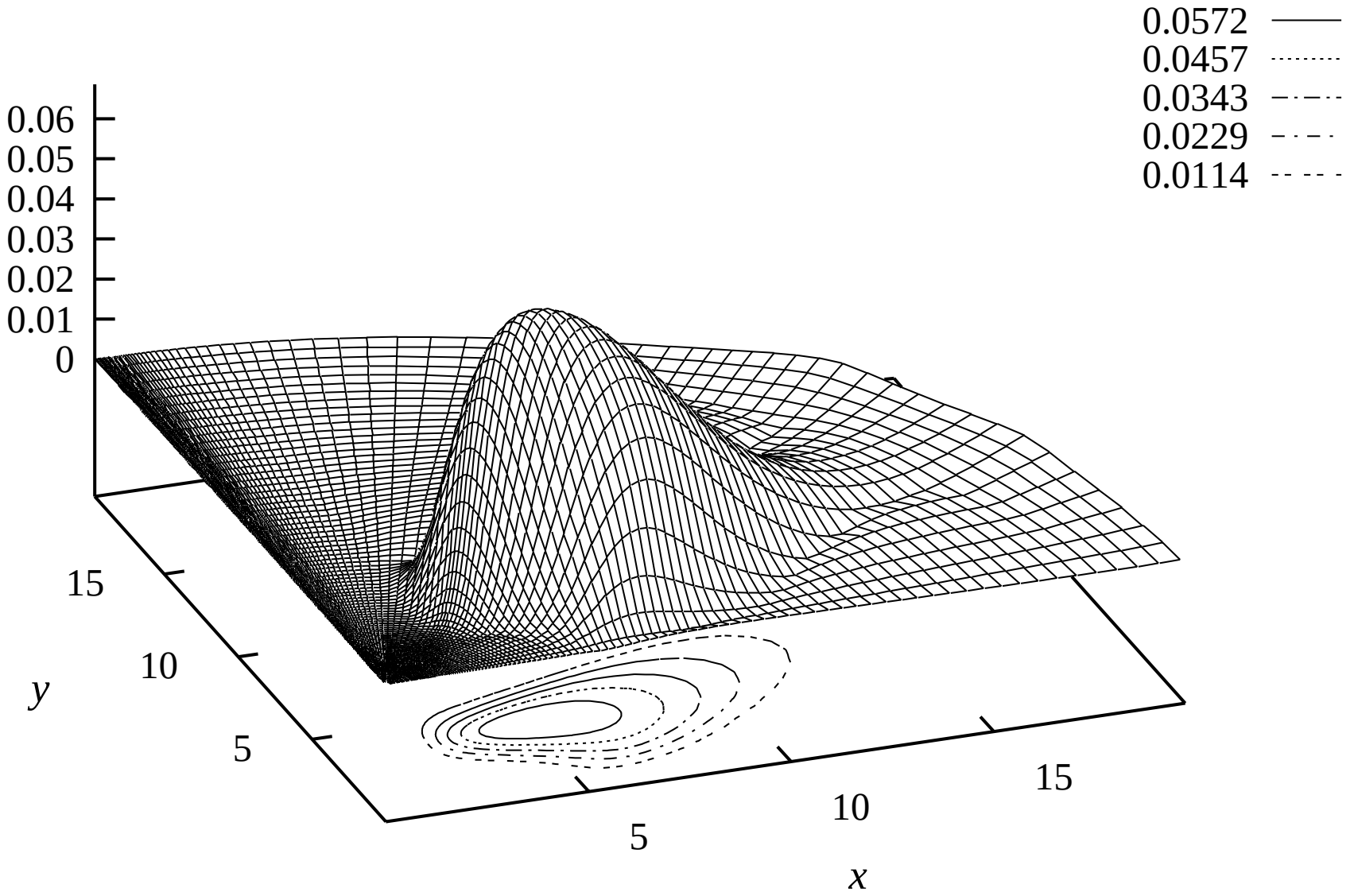
Absolute value of the ^{4}He–^{4}He$_2$ scattering wave function component $\Psi_0(x, y, p)$ for the HFD-B ^{4}He–^{4}He potential at $E = +1.4\,\mathrm{mK}$. **Detail of the surface shown in the previous picture.**



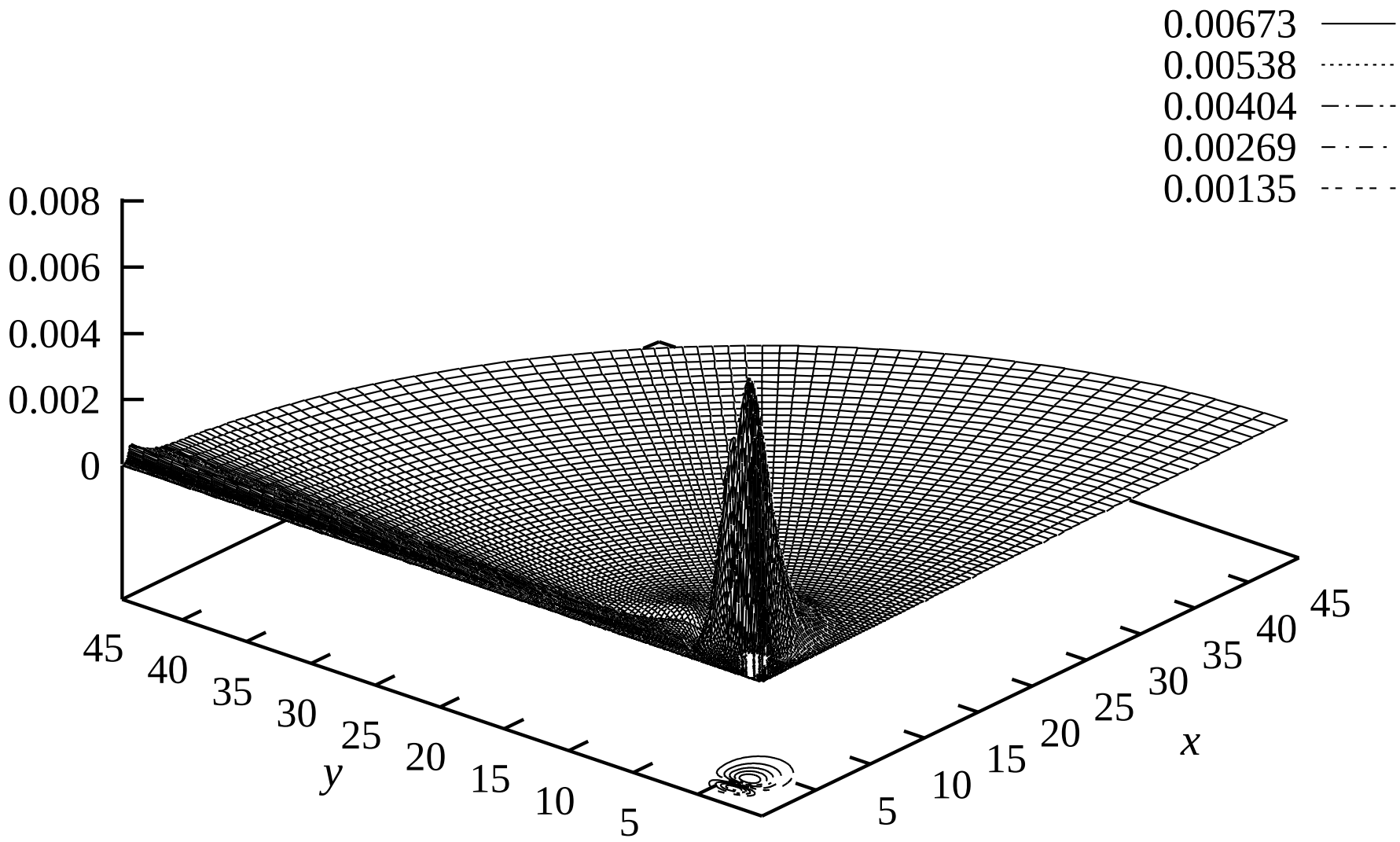
Absolute value of the Faddeev component $\Phi_0(x, y, p)$ for the HFD-B ^{4}He–^{4}He potential at $E = +1.4\,\mathrm{mK}$.



Absolute value of the ^{4}He–^{4}He$_2$ scattering wave function component $\Psi_0(x, y, p)$ for the HFD-B ^{4}He–^{4}He potential at $E = +1.4\,\mathrm{mK}$.



Absolute value of the ^{4}He–^{4}He$_2$ scattering wave function component $\Psi_0(x, y, p)$ for the HFD-B ^{4}He–^{4}He potential at $E = +1.4\,\mathrm{mK}$. **Detail of the surface shown in the previous picture.**



Absolute value of the Faddeev component $\Phi_2(x, y, p)$ for the HFD-B ^{4}He–^{4}He potential at $E = +1.4\,\mathrm{mK}$.

Appendix: Detail formulas

For more details and explanations see [AM, Mathematische Nachrichten **187**, 147 (1997)]

Matrix $\mathbf{M}(z) = \{M_{\alpha\beta}(z)\}, \alpha, \beta = 1, 2, 3$, of the Faddeev components $M_{\alpha\beta}(z)$ acquires on the sheet Π_l the following form

$$\mathbf{M}(z)|_{\Pi_l} = \mathbf{M}(z) - \mathbf{B}^{\dagger}(z)A(z)LS_l^{-1}(z)\tilde{L}\mathbf{B}(z). \qquad (1)$$

Here

$$A(z) = \mathrm{diag}\{A_0(z), A_{1,1}, ..., A_{1,n_3}\},$$

with $A_0(z) = -\pi i z^2$ and $A_{\alpha,j} = -\pi i\sqrt{z - \lambda_{\alpha,j}}$; Π_l:

$$L = \mathrm{diag}\{l_0, l_{1,1}, ..., l_{3,n_3}\} \text{ and } \tilde{L} = \mathrm{diag}\{|l_0|, l_{1,1}, ..., l_{3,n_3}\}$$

where l_0 and $l_{\alpha,j}$ are components of the sheet Π_l multi-index l. By $S_l(z)$ we understand a truncation of the total three-body scattering matrix,

$$S_l(z) = \hat{I} + \tilde{L}[S(z) - \hat{I}]L.$$

At the same time

$$\mathbf{B}(z) = \begin{pmatrix} \mathrm{J}_0\Omega M \\ \mathrm{J}_1\Psi^*[\Upsilon M + \mathbf{v}] \end{pmatrix} \text{ and } \mathbf{B}^{\dagger}(z) = \left(\mathbf{M}(z)\Omega^{\dagger}\mathrm{J}_0^{\dagger}, [\mathbf{v} + M\Upsilon]\Psi\mathrm{J}_1^{\dagger}\right)$$

with

$$\mathbf{v} = \mathrm{diag}\{v_1, v_2, v_3\}, \quad \Omega = (1,\ 1,\ 1), \quad \Upsilon = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } \Psi = \mathrm{diag}\{\Psi_1, \Psi_2, \Psi_3\},$$

where

$$(\Psi_\alpha f)(P) = \sum_{j=1}^{n_\alpha} \psi_{\alpha,j}(k_\alpha) f_j(p_\alpha).$$

The matrix

$$\mathrm{J}_1(z) = \mathrm{diag}\{\mathrm{J}_{1,1}, ..., \mathrm{J}_{3,n_3}(z)\}$$

is combined of the operators $\mathrm{J}_{\alpha,j}(z)$ of restriction on the energy-shells $|p_\alpha|^2 = z - \lambda_{\alpha,j}$.

$J_0 : |P|^2 = z$

$J_{\alpha,j} : |p_\alpha|^2 = z - \lambda_{\alpha,j}$

Representations for $S(z)$:

$$S(z)|_{\Pi_l} = \mathcal{E}(l)\left\{\hat{I} + S_l^{-1}(z)[S(z) - \hat{I}]e(l)\right\}\mathcal{E}(l). \tag{2}$$

Here $\mathcal{E} = \mathrm{diag}\{\mathcal{E}_0, \mathcal{E}_{1,1}, ..., \mathcal{E}_{3,n_3}\}$ where $\mathcal{E}_0$ is identity operator in $L_2(S^5)$ if $l_0 = 0$ and $\mathcal{E}_0$, inversion $(\mathcal{E}_0 f)(\hat{P}) = f(-\hat{P})$ if $l_0 = \pm 1$. Analogously $\mathcal{E}_{\alpha,j}$ are defined in $L_2(S^2)$. $e(l) = \mathrm{diag}\{e_0, e_{1,1}, ..., e_{3,n_3}\}$ with $e_{\alpha,j} = 1$ if $l_{\alpha,j} = 0$ and $e_{\alpha,j} = -1$ if $l_{\alpha,j} = 1$; in any case $e_0 = 1$.

Representations for $R(z) = (H - z)^{-1}$:

$$R(z)|_{\Pi_l} = R + \mathbf{Q}^\dagger A L S_l^{-1} \tilde{L} \mathbf{Q}. \tag{3}$$

Here

$$\mathbf{Q} = \begin{pmatrix} \mathrm{J}_0[I - VR] \\ \mathrm{J}_1\Psi^*[I - \Upsilon M R_0]\Omega^\dagger \end{pmatrix}, \quad \mathbf{Q}^\dagger = \left([I - RV]\mathrm{J}_0^\dagger,\ \Omega[I - R_0 M \Upsilon]\Psi \mathrm{J}_1^\dagger\right)$$

where $V = v_1 + v_2 + v_3$ and $R_0(z) = (H_0 - z)^{-1}$.