Four body integral equations

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The small number of particles in the few-nucleon problem allows accurate solutions of the quantum mechanical many-body problem without the need of approximations, unavoidable for more complex systems. Therefore the comparison of theoretical results with experimental data of can lead to conclusive statements with respect to the assumptions for the underlying nuclear dynamics on which the theory is based.

Technical methods for four-body problems

In the past, several efficient methods have been developed to solve the Schrödinger equation for four-nucleon bound states accurately. These are

- Gaussian-basis variational, the stochastic variational,hyperspherical variational,
- the Green's function Monte Carlo,
- the no-core shell model and
- the effective interaction hyperspherical harmonic methods
- Faddeev-Yakubovsky

2001: benchmark result - seven different approaches obtained the same ^{4}He bound state properties using the Argonne nucleon-nucleon potential
The purpose of these lectures is to provide the pedagodgical introduction into the Faddeev-Yakubovsky method with an example of the four-body problem

History

1956	Skornyakov-Ter-Martirosyan: zero range nuclear forces, divergence for the doublet three nucleon channel, for the first time subdivision of the three nucleon function into (what will be called later) Faddeev components was introduced
1960	Faddeev
60'	Attemts to generalize FE for many-body case: Weinberg (1963) Blankenbecler and Sugar (1964) Rosenberg (1964) and many others
1967	Yakubovsky paper in Sov. J. Nucl. Phys.
1967	Faddeev talk at the Birmingam Few Body Conference
1970	AGS equations
70'	First numerical results for simple separable and local potentials
90'	Beginning of four-body machinery. Solving the FY equations for realistic interactions (W. Glöckle *et al.* Bochum; J. Carbonell *at al.*, Grenoble; Y. Koike it et al., Tokio)

Outline

- Introduction I: Fredholm integral equations.
- Introduction II: Two-body Lippmann-Schwinger equation
- Separable potential. The Hilbert-Schmidt expansion for local interactions
- Faddeev equations
- Four-body equations
- Momentum representation
- Applications: Binding energy of ^{4}He
- Four-body scattering
- Graphical representation
- Applications
- Conclusions

The topics I omit from the discussion

- Other variants of equations for N-body dynamics
- No attempt of mathematical rigor
- Computational methods
- FY equations in the coordinate space
- Inclusion of the Coulomb potential
- Experimental situation in the four-nucleon system is touched only minimally

Introduction I: Fredholm integral equations

The Lippmann-Schwinger (LS) equation represents a particular case of the Fredholm integral equation. Before discussing the LS equation I want to remind a few facts concerning the Fredholm theory. I shall make no attempts at rigor or completeness and shall omit all the proofs which can be found in any standard text book on integral equations and functional analysis.

The inhomogeneous Fredholm equation (IFE) of the second kind is

$$\varphi(x,z) = f(x,z) + \int_0^\infty K(x,s,z)\varphi(s,z)ds,$$

where the parameters z can be everywhere in the complex plain. Assume that the kernel $K(z)$ is the symmetric operator $K(x,s;z) = K(s,x;z)$ and satisfies the Schwartz reflection principle $K(x,s;z^*) = K^*(x,s;z)$ in the complex z plain cut along the real positive semi-axis. We assume also that for all z except the cut

$$\tau(z) = \int_0^\infty \int_0^\infty |K(x,s;z)|^2 dx ds < \infty,$$

so for any complex or negative z operator $K(z)$ is an L^2 or "Hilbert-Schmidt" operator.

Rewrite IFE in the operator form

$$|\varphi(z)\rangle \; = \; |f(z)\rangle + K(z)|\varphi(z)\rangle$$

Any linear integral equation of this type with the L^2 kernel $K(z)$ can be solved immediately if we know the resolvent $\Gamma(z)$

$$|\varphi(z)\rangle = |f(z)\rangle + \Gamma(z)|f(z)\rangle,$$

where $\Gamma(z)$ satisfies

$$\Gamma(z) = K(z) + K(z)\Gamma(z)$$

This equation is a prototype of many integral equations in quantum mechanics and field theory.

Introduction II: Two particle Lippmann-Schwinger equation

Consider non relativistic two-body scattering with Hamiltonian $H = H_0 + V$, H_0 is the kinetic energy operator and V is the interaction potential. All physically interesting information about a system can be obtained if we know the Green function (or the resolvent) $G(z)$, an operator defined for all W outside the spectrum of H by

$$G(z) = (z - H)^{-1}$$

The scattering process may be described in terms of the wave function Ψ^+ which satisfies the Schrödinger equation (E is the energy)

$$H\Psi^+ = E\Psi^+$$

with a certain asymptotic boundary condition in the configuration space. This condition may be roughly formulated as follows

$$\Psi^+ = \Phi \ + \ \text{outgoing wave}$$

The incoming term Φ represents a plane wave for the two particles relative motion. The exact meaning of the term "outgoing wave" may be most simply formulated in the momentum space giving a precise prescription as to how the singularities of the energy denominators are to be dealt with :

$$\Psi^+ = \lim_{\varepsilon \to 0^+} i\varepsilon G(E + i\varepsilon)\Phi$$

To derive the *operator* equation for the Green function we first write

$$\begin{aligned} G(z) \quad &= \quad (z - H_0 - V)^{-1} = [(z - H_0)(1 - G_0(z)V)]^{-1} = \\ & \quad\;\; (1 - G_0(z)V)^{-1} (z - H_0)^{-1} = (1 - G_0 V)^{-1} G_0(z), \end{aligned}$$

where $G_0(z) = (z - H_0)^{-1}$ is the free Green function. Suppose the potential is weak and expand the $(1 - G_0(z)V)^{-1}$ in powers of $G_0 V$.

$$(1 - G_0(z)V)^{-1} = 1 + G_0(z)V + G_0(z)VG_0(z)V + ... \qquad (1)$$

Then for $G(z)$ we obtain the formal iteration series

$$\begin{aligned} G(z) \;&=\; G_0(z) + G_0(z)VG_0(z) + G_0(z)VG_0(z)VG_0(z) + ... \\ &= G_0(z) + G_0(z)V \times [\; G_0(z) + G_0(z)VG_0(z) + \ldots \;] \end{aligned}$$

The expression in square brackets coincides with the iteration series for $G(z)$, therefore

$$G(z) = G_0(z) + G_0(z)VG(z)$$

The above equation represents the (operator) Lippmann-Schwinger equation for the Green function. We stress that these equations are valid even though the iteration series may diverge.

The LS equation can be also written in terms of the T-matrix $T(z)$

$$T = V + VGV, \quad G \;=\; G_0 \;+\; G_0 T G_0,$$

In a perturbation theory an expansion of $T(z)$ in powers of the potential V (assumed to be sufficiently weak) is

$$T(z) = V + VG_0(z)V + VG_0(z)VG_0(z)V + ...$$

The successive terms can be represented graphically.

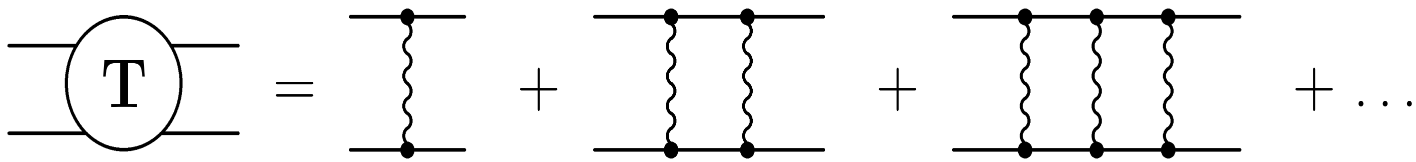


Figure 1: Diagrams for iteration series

Each curly line represents an interaction due to the potential V. In between interactions, the particles move freely (hence the free propagator G_0). Very simple ladder diagrams appear, corresponding to the fact that there are no creation and annihilation phenomena in non-relativistic quantum mechanical problems.

Suppose that we were rash enough to try to calculate $T(z)$ by expanding in powers of V. Using the same trick as was applied to derive the LS equation for $G(z)$, we obtain

$$T(z) = V + VG_0(z)V + VG_0(z)VG_0(z)V + ...$$

$$= V + VG_0(z)[V + VG_0(z)V + VG_0(z)VG_0(z)V + ...] =$$

$$= V + VG_0(z)T(z), \qquad VG = TG_0, \quad GV = G_0T$$

In the momentum space

$$t(\mathbf{k}', \mathbf{k}; z) = V(\mathbf{k}', \mathbf{k}) + \int \frac{V(\mathbf{k}', \mathbf{p})\, t(\mathbf{p}, \mathbf{k}; z)}{z - p^2/2\mu} \frac{d^3p}{(2\pi)^3}$$

The integrand has a fixed singularity at $p = \sqrt{2\,\mu\, z}$. This singularity can be easily treated either analytically or numerically.

In the three-nucleon case break-up boundary condition is reflected by moving singularities. Even so, the numerical treatment is well established.

Since $V(\mathbf{k}', \mathbf{k})$ is the symmetric function, the LS kernel $K(z) = VG_0(z)$ in the energy region below threshold may be reduced to the real symmetric form. Next, for z not on the positive real semi-axis but on the physical sheet, $K(z)$ is a L^2 operator under very general assumptions on V.

$$\tau(z) = \frac{\mu^{\frac{3}{2}}}{2\pi\sqrt{2\,\mathrm{Im}\,z}} \int V^2(r)\, d^3r \;<\; \infty.$$

This assumption for the validity of the Fredholm theory is very weak. In particular, it holds for the potentials usually used in nuclear physics

However, a formal difficulty arises when the energy parameter z takes a positive value, because, no matter how well-behaved the potential is, $\tau(E + i\varepsilon)$ never exist for $E > 0$

This difficulty is only apparent and not real. Here we simply suppose that one can do away with this problem by performing an analytic continuation in z to a region $\mathrm{Im}\,\sqrt{z} \neq 0$. Once we establish the Fregholm properties in this region, we can analytically continue back to physical region of the energy parameter with confidence that the Fredholm properties must still hold. A rigorous proof of this statement is out of scope of the present lectures

The Hilbert identity. Unitarity condition

$$G(z_1) - G(z_2) = (z_2 - z_1)G(z_1)G(z_2)$$

Multiply this equation by V from the left and from the right. In the left-hand side using the relation $VGV = T - V$ we obtain

$$T(z_1) - T(z_2)$$

In the right-hand side using the relations $VG = TG_0$, $GV = G_0T$ we obtain

$$(z_2 - z_1)\, T(z_1)\, G_0(z_1)\, G_0(z_2)\, T(z_2)$$

Let $z_1 = E + i\varepsilon$ and $z_2 = E - i\varepsilon$, then $z_2 - z_1 = -2i\varepsilon$, and

$$T(E + i\varepsilon) - T(E - i\varepsilon) = 2i\mathrm{Im}\, T(E + i\varepsilon) =$$

$$= -2i\,\varepsilon\, T(E + i\varepsilon)\, G_0(E + i\varepsilon)\, G_0(E - i\varepsilon)\, T(E - i\varepsilon)$$

We want to apply this equation to calculate the imaginary part of the on-shell T-matrix $t(\mathbf{k}', \mathbf{k}; z)$ for $z = k^2/2\mu + i\varepsilon = k'^{\,2}/2\mu + i\varepsilon$. We first simplify the notations and define

$$t(\mathbf{k}', \mathbf{p}) = t(\mathbf{k}', \mathbf{p}; k^2/2\mu + i\varepsilon), \quad t(\mathbf{p}, \mathbf{k}) = t(\mathbf{p}, \mathbf{k}; k^2/2\mu + i\varepsilon)$$

$$t(\mathbf{k}', \mathbf{k}) = t(\mathbf{k}', \mathbf{k}; k^2/2\mu + i\varepsilon), \quad |\mathbf{k}'| = |\mathbf{k}|$$

$$\mathrm{Im}\, t(\mathbf{k}', \mathbf{k}) = -\lim_{\varepsilon\to 0} \frac{\varepsilon}{(2\pi)^3} \int d\Omega_{\mathbf{p}} \int \frac{t(\mathbf{k}', \mathbf{p})\, t^*(\mathbf{p}, \mathbf{k})}{(\frac{k^2}{2\mu} - \frac{p^2}{2\mu})^2 + \varepsilon^2}\, p^2 dp$$

Using

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{x^2 + \varepsilon^2} = \pi\delta(x)$$

we get

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\left(\frac{k^2}{2\mu} - \frac{p^2}{2\mu}\right)^2 + \varepsilon^2} = \pi\delta(\frac{k^2}{2\mu} - \frac{p^2}{2\mu}) = 2\mu\pi\delta(k^2 - p^2)$$

Then we obtain

$$\begin{aligned}\mathrm{Im}\, t(\mathbf{k}', \mathbf{k}) &= -\frac{\mu}{4\pi^2} \int d\Omega_{\mathbf{k}'} \int_0^\infty t(\mathbf{k}', \mathbf{p})\, \delta(k^2 - p^2)\, t^*(\mathbf{p}, \mathbf{k}) p^2 dp \\ &= -\frac{\mu}{8\pi^2}\, k \int d\Omega_{\mathbf{k}'}\, |t(\mathbf{k}', \mathbf{k})|^2\end{aligned}$$

For the amplitude $f(\mathbf{k}', \mathbf{k})$ normalized so that $d\sigma/d\Omega = |f(\mathbf{k}', \mathbf{k})|^2$the unitarity condition aquires the familiar form

$$\mathrm{Im} f(\mathbf{k}', \mathbf{k}) = \frac{k}{4\pi} \int d\Omega_{\mathbf{k}'}\, |f(\mathbf{k}', \mathbf{k})|^2$$

The separable potential

The LS equation generally can not be solved analytically. An exception is the so called *separable* (non local) potential in which the dependence on the initial and final momenta is factorized.

$$V(p,p') = -\frac{\lambda}{2\mu}\, g(p)\, g(p')$$

The corresponding T-matrix posses the analytical solution

$$t(p',p;E+i\varepsilon) \;=\; -\frac{\lambda}{2\mu}\, g(p')\, \tau(E+i\varepsilon)\, g(p)$$

with

$$\tau(E+i\varepsilon) \;=\; \left(1 - \frac{\lambda}{2\pi^2}\int_0^\infty \frac{g^2(p)\, p^2\, dp}{k^2 - p^2 + i\varepsilon}\right)^{-1}, \quad k^2 \;=\; m\, E$$

$$g(p) \;=\; \frac{1}{p^2+\beta^2}, \qquad\qquad \tau(z)^{-1} \;=\; 1 - \frac{(\beta+\alpha)^2}{(\beta+\sqrt{-2\mu\, z})^2}$$

HS eigenfunctions and eigenvalues

Define (for a given partial wave)

$$|\, g_m(z)\,\rangle \;=\; \frac{4\pi}{\lambda_m\,(z)}\, V\, G_0(z)\, |g_m(z)\rangle, \qquad g_m(r;E) \;\sim\; \frac{\exp(ikr)}{r}\;\; r \to \infty$$

$$\langle g_m|G_0|g_n\rangle \;=\; -\delta_{mn}, \quad T \;=\; \sum_m |\, g_m\,\rangle\, d_m\, \langle\, g_m\,|, \qquad d_m \;=\; -\,\frac{1}{4\pi}\,\frac{\lambda_m}{1-\lambda_m}$$

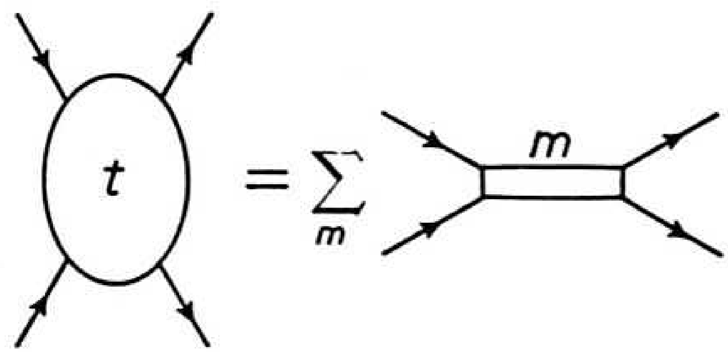


Figure 2: The HS expansion of $T(z)$

Near the pole $z = z_0$

$$\lambda(z) = 1 + \gamma_\lambda (z - z_0), \quad \gamma_\lambda = \left(\frac{d\,\lambda(z)}{d\,z} \right)_{z\,=\,z_0}$$

and

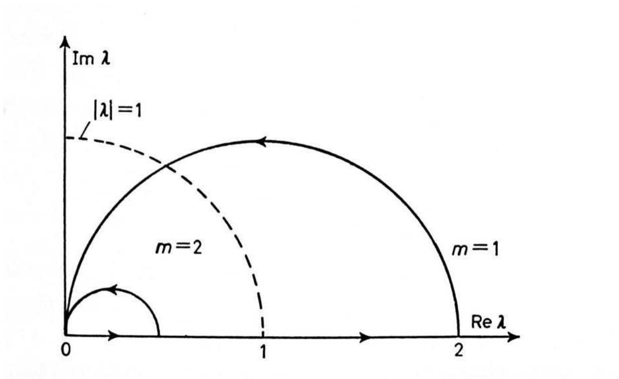
$$T(z) = \frac{4\pi}{\gamma_\lambda} \frac{|g(z_0)\rangle \langle\, g(z_0)|}{z - z_0}$$

Example: the Hulthèn potential $V(r) = -\gamma\,(\exp(r/r_0) - 1)^{-1}$

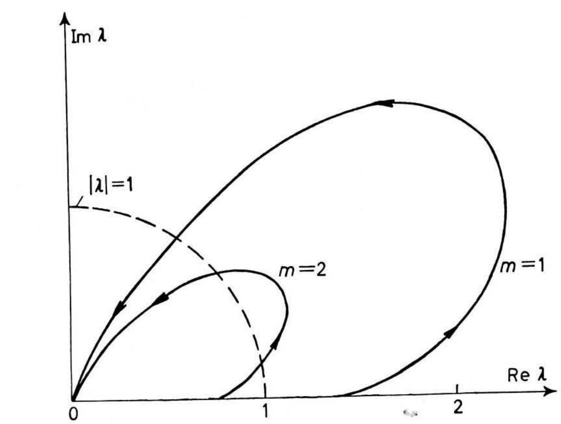
$$\lambda_m(k) = \frac{2\mu\,\gamma\,r_0^2}{m\,(m - 2ikr_0)}$$

For small E

$$\lambda_m(E + i0) - \lambda_m(E) = a_m E + ibE^{l+\frac{1}{2}}$$



a



b

Figure 3: Trajectories of $\lambda(z)$ in the complex plane for some typical attractive potential: s-wave (a), p-wave (b)

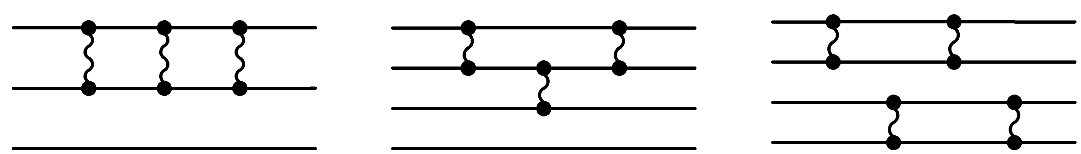
Three body case. Faddeev equations

LS equation

$$T(z) = V + VG_0(z)T(z), \quad V = \sum_{i<j} V_{ij}$$

for more than two particles is not of the type that can be solved directly by the Hilbert-Schmidt method. The trouble can be expressed in a number of ways:

- The kernel $[z - H_0]^{-1}V$ of the LS equation is not of the L^2 type, even if the interactions are well enough behaved to give an L^2 two particle kernel
- The LS kernel contains disconnected graphs



a b c

Figure 4: Disconnected graphs in the three (a) and four-body (b, c) problems. In the last case there are two subsets of disconnected graphs corresponding to partitions (ijk)(l) and (ij)(kl)

The expression for each connected graph contains the overall δ-function expressing conservation of total momentum. This δ-function is completely innocuous, since it can be factored out from the equations. However, any disconnected graph contains the *additional* δ-functions which are not conserved by the full interaction and hence can not be factored out. These dangerous δ-functions do not disappear after iterations and lead to the fact that the kernels are not of L^2 type.

Without making no attempts at mathematical rigor, we consider below the absence of δ-functions or their removing after iterations as the formal criterium for completely continuous kernels. We can now formulate a problem:

> Can we rewrite the LS equations as a set of linear integral equations with the kernels which are either connected or become connected after some number of iterations?

We suppose that the interactions behave well enough to give a L^2 two-particle kernel

Define

$$T^{\alpha} = V_{\alpha} + V_{\alpha} G_0(z) T(z), \quad \alpha = 12, 31, 23, \qquad T(z) = \sum_{\alpha} T^{\alpha}(z)$$

These operators satisfy the equation $T^{\alpha} G_0 = V_{\alpha} G$

$$T^{\alpha} = V_{\alpha} + V_{\alpha} G_0(z) \sum_{\beta} T^{\beta}(z), \quad \beta = 12, 31, 23$$

Rearranging terms, one finds

$$\left(1 - V_{\alpha} G_0(z)\right) T^{\alpha} = V_{\alpha} + V_{\alpha} G_0(z) \sum_{\beta \neq \alpha} T^{\beta}(z),$$

We multiply this equation by $\left(1 - V_{\alpha} G_0(z)\right)^{-1}$ from the left. Using

$$\left(1 - V_{\alpha} G_0(z)\right)^{-1} V^{\alpha} = T_{\alpha},$$

one obtains the Faddeev equations

$$T^{\alpha} = T_{\alpha} + T_{\alpha} G_0(z) \sum_{\beta \neq \alpha} T^{\beta}(z),$$

$T^{\alpha}(z)$ is the three body operator, while $T_{\alpha}(z)$ is the two-body operator. Both are defined in the three body Hilbert space

The kernel of FE become connected after one iteration:

$$T^{12}(z) = T_{12}(z) + T_{12}G_0(z)T_{31}(z) + T_{12}(z)G_0(z)T_{23}(z) + ...$$

The equivalence of Eqs.

$$T^{\alpha} = T_{\alpha} + T_{\alpha}G_0(z)\sum_{\beta\neq\alpha} T^{\beta}(z),$$

and

$$T^{\alpha} = V_{\alpha} + V_{\alpha}G_0(z)T(z)$$

follows from the fact that in the Faddeev methods all the transformations are reversible. It may be not the case for other variants of many-body equations suggested in the literature (P.Federbush (1966), C.Chandler (1978)).

The simple origin of the spurious solutions to a homogeneous equation $\psi = \mathbb{K}\psi$ is the factorization property of the kernel $1 - \mathbb{K}$ (R. Newton, 1966). We demonstrate the factorization with an example of the 3-body Weinberg equation

$$\psi = G_0 \sum_\alpha V_\alpha \tilde{G}_\alpha V^\alpha \psi, \;\; V^\alpha = V - V_\alpha$$

$$\tilde{G}_\alpha = G_\alpha - G_0 = G_0 V_\alpha G_\alpha$$

It can be easily checked that

$$\left(\mathbb{I} - G_0 \sum_\alpha V_\alpha G_0 V^\alpha\right) = \left(\mathbb{I} + \sum_\alpha G_0 V_\alpha\right)\left(\mathbb{I} - G_0 \sum_\alpha V_\alpha\right)$$

Thus we have

$$\left(\mathbb{I} + \sum_\alpha G_\alpha V_\alpha\right)\left(\mathbb{I} - \sum_\alpha V_\alpha\right) = 0$$

Therefore the Weinberg equation may have solutions of two types. The first solutions satisfy the Shrödinger¨ equation

$$\left(\mathbb{I} - G_0 \sum_\alpha V_\alpha \right) = 0$$

at those energies at which $H = H_0 + V$ has an eigenvalue. Besides, solutions may arise defined by

$$\left(\mathbb{I} - G_\alpha \sum_\alpha V_\alpha \right) \psi = \chi \neq 0$$

with

$$\left(\mathbb{I} + \sum_\alpha G_\alpha V_\alpha \right) = 0$$

These solutions are not predicted by the Shrödinger theory and usually are called *spurious solutions*

A factorization property does not guarantee the existence of spurious solutions, however it makes their existence rather probable. In particukar Federbush has suggested a very special model for which the homogeneous part of Weinberg equations admits the spurious solutions in the first sheet of the complex energy plane. Chandler has shown the existence of spurious solutions in the Federbush model for many N-particle integral equations suggested in the literature.

Operators T_α are left-classified. In the four body problem, at least the kernels are classified from both sides. Define

$$M_a^{\alpha,\beta} = V_\alpha \delta_{\alpha\beta} + V_\alpha G_a V_\beta, \quad a = (ijk)(l) \ \text{ or } \ (ij)(kl)$$

The operators $M_a^{\alpha,\beta}$ satisfy

$$M_a^{\alpha,\beta} = T_\alpha \delta_{\alpha\beta} + T_\alpha G_0 \sum_{(\gamma\neq\alpha)\subset a} M_a^{\gamma,\beta}$$

These equations can be written in the matrix form

$$\begin{pmatrix} M^{12,12} & M^{12,31} & M^{12,23} \\ M^{31,12} & M^{31,31} & M^{31,23} \\ M^{23,12} & M^{23,31} & M^{23,23} \end{pmatrix} = \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} +$$

$$\begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} \begin{pmatrix} 0 & G_0 & G_0 \\ G_0 & 0 & G_0 \\ G_0 & G_0 & 0 \end{pmatrix} \begin{pmatrix} M^{12,12} & M^{12,31} & M^{12,23} \\ M^{31,12} & M^{31,31} & M^{31,23} \\ M^{23,12} & M^{23,31} & M^{23,23} \end{pmatrix}$$

Rewrite the last equation in the form

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} \begin{pmatrix} 0 & G_0 & G_0 \\ G_0 & 0 & G_0 \\ G_0 & G_0 & 0 \end{pmatrix}\right] \times$$

$$\times \begin{pmatrix} M^{12,12} & M^{12,31} & M^{12,23} \\ M^{31,12} & M^{31,31} & M^{31,23} \\ M^{23,12} & M^{23,12} & M^{23,23} \end{pmatrix} = \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix}$$

hence

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} \begin{pmatrix} 0 & G_0 & G_0 \\ G_0 & 0 & G_0 \\ G_0 & G_0 & 0 \end{pmatrix}\right]^{-1} \times$$

$$\times \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} = \begin{pmatrix} M^{12,12} & M^{12,31} & M^{12,23} \\ M^{31,12} & M^{31,31} & M^{31,23} \\ M^{23,12} & M^{23,12} & M^{23,23} \end{pmatrix}$$

This identity will be used in the derivation of the four-body equations

$\langle \mathbf{k}, \mathbf{p} \,| M^{\alpha\beta}(z) |\, \mathbf{k}, \mathbf{p} \rangle = \mathcal{M}^{(1)}(\mathbf{k}, \mathbf{p}; \mathbf{k}', \mathbf{p}'; z)$ $(\alpha = \beta)$ or $\mathcal{M}^{(2)}(\mathbf{k}, \mathbf{p}; \mathbf{k}', \mathbf{p}'; z)$ $(\alpha \neq \beta)$

The function $\mathcal{M}^{(1)}$ contains a term of the form $t(\mathbf{k}, \mathbf{k}'; z_p)\, \delta(\mathbf{p} - \mathbf{p}')$

$$\tilde{\mathcal{M}}^{(1,2)}(k', \mathbf{p}';\, k, \mathbf{p}; z) \;= \qquad\qquad (z_p = z - \frac{3}{4}\frac{p^2}{m},\; z_{p'} = z - \frac{3}{4}\frac{p'^2}{m})$$

$$= \sum_{mm'} g_{m'}(k'; z_{p'})\, d_{m'}(z_{p'})\, a^{(1,2)}_{m'm}(\mathbf{p}', \mathbf{p}; z)\, d_m(z_p)\, g_m(k; z_p)$$

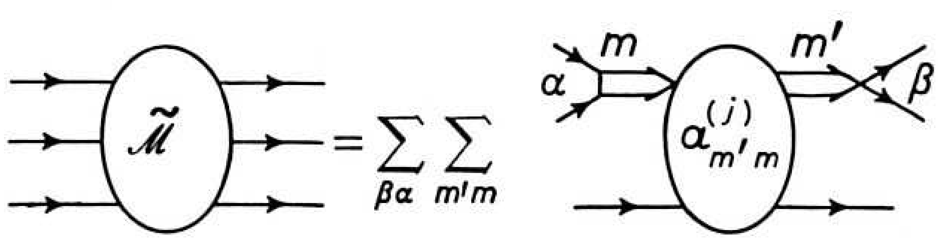


Figure 5: Definition of $a^{(1)}_{m'm}(\mathbf{p}', \mathbf{p}; z)$ $(\alpha = \beta)$ and $a^{(2)}_{m'm}(\mathbf{p}', \mathbf{p}; z)$ $(\alpha \neq \beta)$

$$a_{m'm}(\mathbf{p}', \mathbf{p}; z) = 2\,\mathbb{U}_{m'm}(\mathbf{p}', \mathbf{p}; z) +$$

$$+\,2 \sum_{m''} \int \mathbb{U}_{m'm''}(\mathbf{p}', \mathbf{p}''; z)\, d_{m''}(z_{p''})\, a_{m''m}(\mathbf{p}'', \mathbf{p}; z)\, d^3p''$$

$$\mathbb{U}_{m'm}(\mathbf{p}', \mathbf{p}; z) = \frac{g_{m'}(p_1; z_{p'})\, g_m(p_2; z_p)}{z - \frac{1}{m}(p^2 + p'^2 + \mathbf{p}\mathbf{p}')}$$

$$p_1 = |\mathbf{p} + \frac{1}{2}\mathbf{p}'|, \;\; p_2 = |\frac{1}{2}\mathbf{p} + \mathbf{p}'|, \quad d_m(z) = -\frac{1}{4\pi}\,\frac{\lambda_m(z)}{1 - \lambda_m(z)}$$

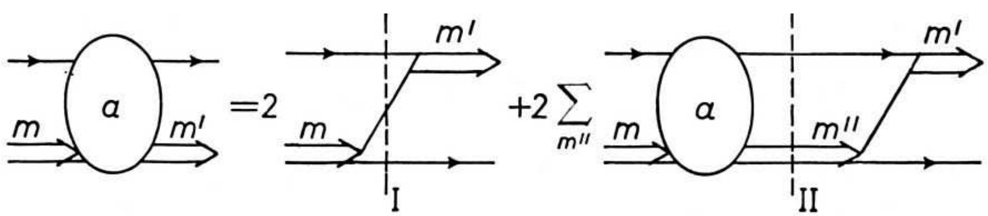


Figure 6: Faddeev equations for $a_{m'm}(z) = a^{(1)}_{m'm}(z) + 2\,a^{(2)}_{m'm}(z)$. The elastic scattering nd amplidude is $F_{nd} = -\frac{\pi\mu_{nd}}{\gamma_1}\, a_{11}(\mathbf{p}', \mathbf{p};\, E + i0)$

Since equation for the amplitudes $a_{m'm}(z)$ has the resolvent form we can apply again the HS method. For a given 3-body partial wave we define the 3-body eigenfunctions and eigenvalues

$$|w_n(z)\rangle = \frac{8\pi}{\eta_n(s)} \mathbb{U}(z)\, d(z_p)\, |w_n(z)\rangle$$

$$|w_n z\rangle = \begin{pmatrix} w_{n1} \\ w_{n2} \\ w_{n3} \\ \dots \end{pmatrix} \qquad \mathbb{U} = \begin{pmatrix} \mathbb{U}_{11} & \mathbb{U}_{12} & \mathbb{U}_{13} & \cdots \\ \mathbb{U}_{21} & \mathbb{U}_{22} & \mathbb{U}_{23} & \cdots \\ \mathbb{U}_{31} & \mathbb{U}_{32} & \mathbb{U}_{33} & \cdots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

$$a_{mm'}(p', p; z) = \sum_n g_{m'n}(p'; z)\, \omega_n(z)\, g_{mn}(p; z), \quad \omega_n(z) = -\frac{1}{4\pi} \frac{\eta_n(z)}{1 - \eta_n(z)}$$

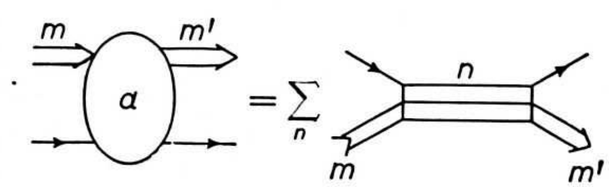


Figure 7: HS expansion for the three-body amplitude

Four body integral equations

Consider one of the Faddeev equations for four particles

$$T^{12} = T_{12} + T_{12}G_0(T^{31} + T^{23} + T^{41} + T^{42} + T^{34})$$

T^{α} $(\alpha = 12,\ 31,\ 23,\ 41,\ 42,\ 34)$ are four-body operators, T_{12} is the two-body operator. Iterations of these equations still contain disconnected graphs

$$\begin{aligned} T^{12} \quad &= \quad T_{12} + T_{12}G_0T_{34} + T_{12}G_0T_{34}G_0T_{12} + \ldots \quad b = (12)(34) \\ &\quad + T_{12}G_0T_{31} + T_{12}G_0T_{31}G_0T_{23} + \ldots \qquad a = (123)(4) \end{aligned}$$

Let us introduce three operators

$$T^{12,123} = T_{12}G_0(T^{31} + T^{23}), \quad T^{12,124} = T_{12}G_0(T^{41} + T^{42}), \quad T^{12,34} = T_{12}G_0T^{34}$$

$$T^{\alpha a} = T_{\alpha}G_0 \sum_{(\beta \neq \alpha) \subset a} T^{\beta}$$

It is evident that

$$T^{12} = T_{12} + T^{12,123} + T^{12,124} + T^{12,34}$$

In general case equations

$$T^{12,123} = T_{12} G_0 (T^{31} + T^{23}), \quad T^{12,124} = T_{12} G_0 (T^{41} + T^{42}),$$

$$T^{12,34} = T_{12} G_0 T^{34}$$

are written as

$$T^{\alpha a} = T_\alpha G_0 \sum_{(\beta \neq \alpha) \subset a} T^\beta$$

Substitute for T^β

$$T^\beta = T_\beta + \sum_{(b \supset \beta)} T^{\beta b}$$

Then we obtain

$$T^{\alpha a} = T_\alpha G_0 \sum_{(\beta \neq \alpha) \subset a} T_\beta \quad + \quad T_\alpha G_0 \sum_{(\beta \neq \alpha) \subset a} \sum_{b \supset \beta} T^{\beta b}$$

or, after rearranging terms,

$$T^{\alpha a} - T_\alpha G_0 \sum_{(\beta \neq \alpha) \subset a} T^{\beta a} = T_\alpha G_0 \sum_{(\beta \neq \alpha) \subset a} T_\beta + T_\alpha G_0 \sum_{(b \supset \beta) \neq a} T^{\beta b}$$

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} \begin{pmatrix} 0 & G_0 & G_0 \\ G_0 & 0 & G_0 \\ G_0 & G_0 & 0 \end{pmatrix}\right] \begin{pmatrix} T^{12,123} \\ T^{31,123} \\ T^{23,123} \end{pmatrix} =$$

$$= \begin{pmatrix} T_{12}\ G_0\ (T_{31} + T_{23}) \\ T_{31}\ G_0\ (T_{12} + T_{23}) \\ T_{23}\ G_0\ (T_{12} + T_{31}) \end{pmatrix} +$$

$$+ \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} \begin{pmatrix} 0 & G_0 & G_0 \\ G_0 & 0 & G_0 \\ G_0 & G_0 & 0 \end{pmatrix} \begin{pmatrix} \sum\limits_{b \neq (123)(4)} T^{12,b} \\ \sum\limits_{b \neq (123)(4)} T^{31,b} \\ \sum\limits_{b \neq (123)(4)} T^{23,b} \end{pmatrix}$$

The condition $b \supset \beta$ is understood:

$$T^{12,b} = T^{12,124}, | \, T^{12,34}, \quad T^{31,b} = T^{31,314}, \ T^{31,23}, \quad T^{23,b} = T^{23,234}, \ T^{23,14}$$

Recall that in the derivation of the three body equations we had to calculate

$$(1 - V_\alpha G_0(z))^{-1} V^\alpha = T_\alpha,$$

In the four-body case we have to calculate

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} \begin{pmatrix} 0 & G_0 & G_0 \\ G_0 & 0 & G_0 \\ G_0 & G_0 & 0 \end{pmatrix}\right]^{-1} \times$$

$$, \qquad \times \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix}$$

In the lecture on the three body equations we have already derived

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} \begin{pmatrix} 0 & G_0 & G_0 \\ G_0 & 0 & G_0 \\ G_0 & G_0 & 0 \end{pmatrix}\right]^{-1} \times$$

$$\times \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} = \begin{pmatrix} M^{12,12} & M^{12,31} & M^{12,23} \\ M^{31,12} & M^{31,31} & M^{31,23} \\ M^{23,12} & M^{23,12} & M^{23,23} \end{pmatrix}$$

In the same way one derive

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} T_{12} & 0 & 0 \\ 0 & T_{31} & 0 \\ 0 & 0 & T_{23} \end{pmatrix} \begin{pmatrix} 0 & G_0 & G_0 \\ G_0 & 0 & G_0 \\ G_0 & G_0 & 0 \end{pmatrix}\right]^{-1} \times$$

$$\times \begin{pmatrix} T_{12}\; G_0\; (T_{31} + T_{23}) \\ T_{31}\; G_0\; (T_{12} + T_{23}) \\ T_{23}\; G_0\; (T_{12} + T_{31}) \end{pmatrix} = \begin{pmatrix} \tilde{T}_a^{12} \\ \tilde{T}_a^{31} \\ \tilde{T}_a^{23} \end{pmatrix}, \quad a = (123)(4)$$

$$\begin{aligned} T^{12,123} = \tilde{T}_a^{12} \quad + \quad & (M_a^{12,31} + M_a^{12,23})\; G_0\; (T^{12,124} + T^{12,34}) \;+ \\ & (M_a^{12,12} + M_a^{12,23})\; G_0\; (T^{31,134} + T^{31,24}) \;+ \\ & (M_a^{12,12} + M_a^{12,31})\; G_0\; (T^{23,234} + T^{23,34}) \end{aligned}$$

$$\begin{aligned} T^{12,34} = \tilde{T}_b^{12} \quad + \quad & N_b^{12,34}\; G_0\; (T^{13,123} + T^{12,124}) \;+ \\ & N_b^{12,12}\; G_0\; (T^{34,134} + T^{34,234}), \end{aligned}$$

$$a = (123)(4), \quad b = (12)(34)$$

The general form of the four-body FY equations:

$$T^{\alpha a} = \tilde{T}_a^{\alpha} + \sum_{(\gamma\neq\delta)\subset a} \sum_{d\supset\delta,\, d\neq a} M_a^{\alpha,\gamma} G_0\, T^{\delta d}$$

In contrast to the Weinberg equations, the kernels $M_a^{\alpha,\beta}$ are not connected or almost connected ones. Indeed the diagonal components $M_a^{\alpha,\beta}$ contain the pair T-matrices T_α. However, the δ-functions corresponding to disconnected graphs are removed after two iterations.

The FY system of equations is of the Fredholm type and has an unique solution (K.Hepp (1969))

Bound state equations

$$\Psi^{\alpha a} = \sum_{(\gamma\neq\delta)\subset a} \sum_{d\supset\delta,\, d\neq a} M_a^{\alpha,\gamma} G_0\, \Psi^{\delta d}$$

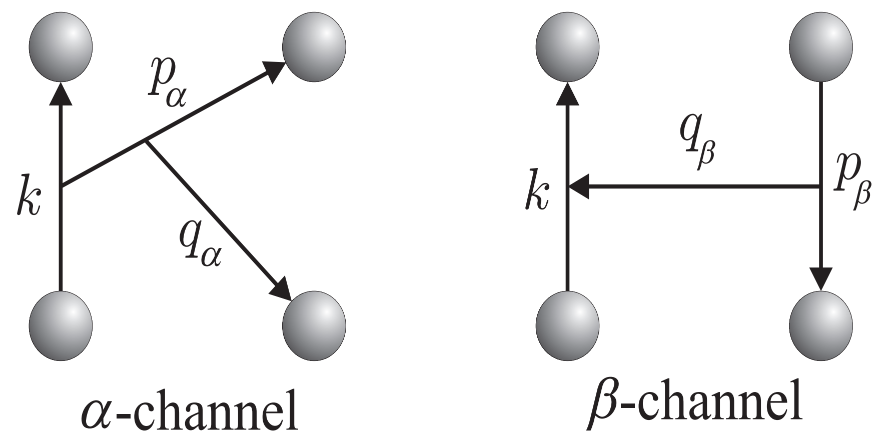


Figure 8: Jacobi momenta corresponding to the different partitions. α channel: partition a=(ijk)(l), β channel: partition b=(ij)(kl)

$$\mathbf{k}_{ij} = \frac{\mathbf{p}_i - \mathbf{p}_j}{2}, \quad \mathbf{p}_{ij,k} = \frac{\mathbf{p}_i + \mathbf{p}_j - 2\mathbf{p}_k}{3}, \quad \mathbf{q}_{ijk,l} = \frac{\mathbf{p}_i + \mathbf{p}_j + \mathbf{p}_k - 3\mathbf{p}_l}{4} \qquad (\alpha)$$

$$\mathbf{k}_{ij} = \frac{\mathbf{p}_i - \mathbf{p}_j}{2}, \quad \mathbf{p}_{kl} = \frac{\mathbf{p}_k - \mathbf{p}_l}{2}, \quad \mathbf{q}_{ij,kl} = \frac{\mathbf{p}_i + \mathbf{p}_j - \mathbf{p}_k - \mathbf{p}_l}{4} \qquad (\beta)$$

Jacobi momenta separate the center of mass motion and guarantee a KE operator independent from angular variables. Other coordinates lead to strong angular dependence and, as a result, to a very slowly converging series of partial waves.

In a system of identical particles

$$\Psi^{ij,ijk}(\mathbf{k}_{ij}, \mathbf{p}_{ijk}, \mathbf{q}_l) = \Psi_1(\mathbf{k}_{ij}, \mathbf{p}_{ijk}, \mathbf{q}_l)$$

$$\Psi^{ij,kl}(\mathbf{k}_{ij}, \mathbf{p}_{kl}, \mathbf{q}_{ij}) = \Psi_2(\mathbf{k}_{ij}, \mathbf{p}_{kl}, \mathbf{q}_{ij})$$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{K}_{21} & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

Because a state must be even under the permutation of each two particles and of two-particle clusters

$$\Psi_1(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \Psi_1(-\mathbf{k}, \mathbf{p}, \mathbf{q})$$

$$\Psi_2(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \Psi_2(-\mathbf{k}, \mathbf{p}, \mathbf{q}) = \Psi_2(\mathbf{k}, -\mathbf{p}, \mathbf{q}) = \Psi_2(-\mathbf{p}, \mathbf{k}, -\mathbf{q})$$

The integral operators $M^{\alpha,\beta}_{(ijk)(l)}$ are expressed in terms of two functions

$$\begin{aligned}\langle \mathbf{k},\mathbf{p},\mathbf{q}\,|M^{\alpha\beta}(z)|\,\mathbf{k},\mathbf{p},\mathbf{q}\rangle \quad &= \quad \mathcal{M}^{(1)}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z-\frac{2}{3}\;q^2)\;\delta(\mathbf{q}-\mathbf{q}'), \qquad \alpha=\beta \\ & \quad\;\; \mathcal{M}^{(2)}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z-\frac{2}{3}\;q^2)\;\delta(\mathbf{q}-\mathbf{q}'), \qquad \alpha\neq\beta\end{aligned}$$

which are even in $\mathbf{k}$, $\mathbf{k}'$.
Analogously $N^{\alpha,\beta}_{(ij)(kl)}$ are expressed in terms of $\mathcal{N}^1$ and $\mathcal{N}^2$

$$\begin{aligned}\langle \mathbf{k},\mathbf{p},\mathbf{q}\,|N^{\alpha\beta}(z)|\,\mathbf{k},\mathbf{p},\mathbf{q}\rangle \quad &= \quad \mathcal{N}^{(1)}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z-\frac{1}{2}\;q^2)\;\delta(\mathbf{q}-\mathbf{q}'), \qquad \alpha=\beta \\ & \quad\;\; \mathcal{N}^{(2)}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z-\frac{1}{2}\;q^2)\;\delta(\mathbf{q}-\mathbf{q}'), \qquad \alpha\neq\beta\end{aligned}$$

The functions $\mathcal{M}^{(1)}$ and $\mathcal{N}^{(1)}$ contain a term with $\delta(\mathbf{p}-\mathbf{p}')$ of the form

$$t\;(\mathbf{k},\;\mathbf{k}';z-\frac{3}{4}\;q^2)\;\delta(\mathbf{p}-\mathbf{p}')\;\delta(\mathbf{q}-\mathbf{q}')$$

or

$$t\;(\mathbf{k},\;\mathbf{k}';z-\frac{1}{2}\;q^2)\;\delta(\mathbf{p}-\mathbf{p}')\;\delta(\mathbf{q}-\mathbf{q}')$$

This δ-function is removed after two iterations

We use the Dirac delta-functions to perform one of the momentum integrals, change some variables and the symmetry properties of Ψ_1 and Ψ_2 to get

$$\Psi_1(\mathbf{k},\mathbf{p},\mathbf{q}) = \left(E_B - k^2 - \frac{3}{4}p^2 - \frac{2}{3}q^2\right)^{-1} \times$$
$$\left(\int \mathcal{M}^s\ (\mathbf{k},\mathbf{p};\frac{1}{2}\ \mathbf{p}' + \mathbf{K_1},\mathbf{p}';\ E_B - \frac{2}{3}q^2)\ \Psi_1(\mathbf{p}' + \frac{1}{2}\ \mathbf{K_1},\ \mathbf{K_2},\mathbf{q}')\ d\mathbf{p}' d\mathbf{q}' +\right.$$
$$\left.\eta^P \int \mathcal{M}^s\ (\mathbf{k},\mathbf{p};\ \frac{1}{2}\ \mathbf{p}' +\ \mathbf{P}_1,\mathbf{p}';\ E_B - \frac{2}{3}q^2)\ \Psi_2(\mathbf{p}' + \frac{1}{2}\ \mathbf{P}_1,\ \mathbf{Q}_2,\mathbf{q}')\, d\mathbf{p}' d\mathbf{q}'\ \right)$$

$$\Psi_2(\mathbf{k},\mathbf{p},\mathbf{q}) \quad = \quad \left(E_B - k^2 - p^2 - \frac{1}{2}q^2\right)^{-1} \times$$
$$2\int \mathcal{N}^s\ (\mathbf{k},\mathbf{p};Q_1,\ \mathbf{p}';\ E_B - \frac{1}{2}q^2)\,\Psi_1(\mathbf{p},\ \mathbf{P}_2,\ \mathbf{q}')\,d\mathbf{p}' d\mathbf{q}'$$

$$\mathbf{K_1} = \frac{1}{3}\mathbf{q} + \mathbf{q}' \quad \mathbf{K_2} = \mathbf{q} + \frac{1}{3}\mathbf{q}', \quad \mathbf{P_1} = -\frac{2}{3}\mathbf{q} - \mathbf{q}' \quad \mathbf{P_2} = -\mathbf{q} - \frac{1}{3}\mathbf{q}',$$
$$\mathbf{Q_1} = \frac{1}{2}\mathbf{q} + \mathbf{q}', \quad \mathbf{Q_2} = \mathbf{q} - \frac{1}{2}\mathbf{q}'$$

The integral kernels are

$$\mathcal{M}^s(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z) = \mathcal{M}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z) + \mathcal{M}(-\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z),$$

$$\mathcal{M}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z) = \mathcal{M}^{(1)}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z) + 2\mathcal{M}^{(2)}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z),$$

$$\mathcal{N}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z) = \mathcal{N}^{(1)}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z) + \eta^P\mathcal{N}^{(2)}(\mathbf{k},\mathbf{p};\mathbf{k}',\mathbf{p}';z)$$

At present, a variety of NN-potentials is available which fit the NN data with magnificent accuracy. Beyond the longest range OPE part, the medium and short range region is parametrized purely phenomenologically by the exchange of heavier mesons and introduction of form factors. Modern NN-forces contain typically 40-50 parameters but describe NN scattering data with high precision up to 350 MeV.

Dictionary

- ✦ Separable potential with the dipole (Y) or exponential (E) form factors
- ✦ MT I-III, s wave phenomenological Malfliet-Tjon potential, contains two Yukava terms, corresponding to the long range attraction and short-range repulsion, R.A. Malfliet, J. Tjon, Nucl. Phys. **A127** 161 (1969)
- ✦ Argonne potential AV18, R.B.Wiringa *et al.* PRC, **51** 38 (1995). The AV8$'$ interaction is derived from the AV18 interaction by neglecting the charge dependence and the terms proportional to L^2 and $(L \cdot S)^2$. This potential consists of 8 parts (V_i are radial functions of Yukawa- and Wood-Saxon types):

$$\begin{aligned} V(r) &= V_c(r) + V_\tau(r)(\tau \cdot \tau) + V_\sigma(r)(\sigma \cdot \sigma) + V_{\sigma\tau}(r)(\sigma \cdot \sigma)(\tau \cdot \tau) \\ &\quad + V_t(r) S_{12} + V_{t\tau}(r) S_{12}\ (\tau \cdot \tau) \\ &\quad + V_b(r)(L \cdot S) + V_{b\tau}(r)(L \cdot S)\ (\tau \cdot \tau) = \sum_{i=1}^{8} V_i(r) \mathcal{O}_i, \end{aligned}$$

- ✦ Bonn potential CD Bonn (2000), R. Machleidt PRC **63** 024001 (2000), Based on meson exchange

Early calculations of the α-particle BE

Potential	Method	B_t	B_α	B_{α^*}	References
Separable Y	HS	-11.03	-45.73	-11.69	ITEP (1973)
Separable Y	2-dimensional eqs	-11.05	-45.70		GL (1976)
Separable Y	Bateman	-10.43	-45.18	10.88	ITP (1976)
Separable E	-9.82	–39.83	-25.90	-10.08	ITP (1976)
Separable G	2-dimensional eqs	-8.99	-33.3		KS (1978)
MT	HS	-8.56	-29.6		Tjon (1975)
RSC	HS	-6.8	-25.80		Tjon(1978)

ITEP: IMN, E.S.Galpern, and V.N.Lyakhovitsky, Phys. Lett. **B46** 51 (1973)
GL: B.Gibson, D.R.Lehman, Phys. Rev. **C14** 685 (1976)
ITP: V.F.Kharchenko and V.P.Levashev, Phys. Lett. **B60** 317 (1976)
KS: H.Kröger, W. Sandhas: Phys. Rev. Lett. **40** 834 (1978)
Tjon: J.Tjon, Phys. Lett. **B50** 217 (1975), Phys. Rev. Lett. **B40** 1239 (1978)

The expectation values $\langle T \rangle$ and $\langle V \rangle$ of kinetic and potential energies, the binding energies E_b in MeV and the radius in fm

Method	$\langle T \rangle$	$\langle V \rangle$	E_b	$\sqrt{\langle r^2 \rangle}$
FY	102.39	-128.33	-25.94	1.485
CRCGV	102.30	-128.20	-25.90	1.482
SVM	102.35	-128.27	-25.92	1.486
HH	102.44	-128.34	-25.90	1.483
GFMC	102.30	-128.25	-25.93	1.490
NCSM	103.35	-129.45	-25.80	1.485
EIHH	100.78	-126.72	-25.94	1.486

- ✦ CRCGV: Coupled-rearrangement-channel Gaussian-basis variational
- ✦ SVM: The stochastic variational
- ✦ HH: The hyperspherical variational
- ✦ GFMC: The Green's function Monte Carlo
- ✦ NCSM:The no-core shell model
- ✦ EIHH: The effective interaction hyperspherical harmonic method

^4He binding energy

	BE (MeV)			P_D (%)		R (fm)	
Method	HH	FY	NCSM	HH	FY	HH	FY
AV18	24.22	24.25		13.74	13.78	1.512	1.516
CD-Bonn	26.13	26.16		10.74	10.77	1.454	
N3LO	25.38	25.37	25.36	9.29	9.29	1.516	

FY: Nogga et al, PRC 65, 054003 (2002)

NCSM: Navratil & Barret, PRC 59, 014311 (2004)

HH: MV, Marcucci, Kievsky & Rosati, 2005

Figure 9:

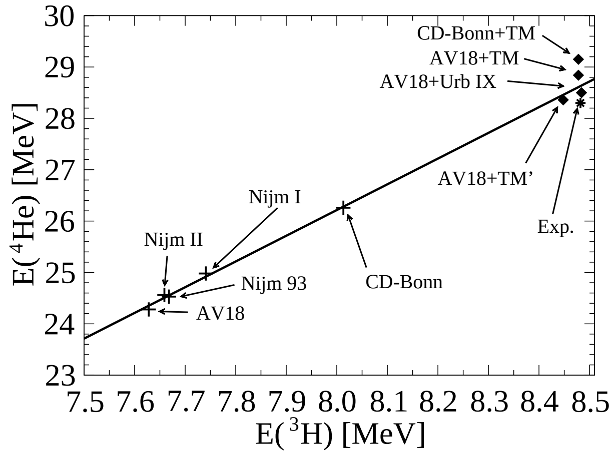


Figure 10: The modern Tjon line: α particle vs ^{3}H binding energy

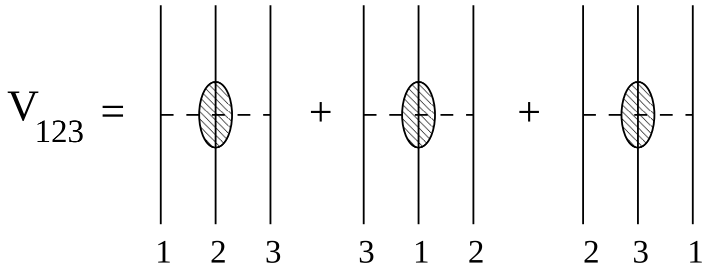


Figure 11: Three nucleon interaction

Scattering problem. Doubly classified four-body operators

Define (O.A.Yakubovsky, PhD thesis, 1967)

$$W^{\alpha a,\beta b} = T_\alpha G_0 T_\beta (1-\delta_{\alpha\beta}) + \sum_{\gamma\neq\ \alpha,\,\gamma\subset a}\ \sum_{(\delta\neq\beta)\subset a} T_\alpha G_0 M^{\gamma\beta} G_0 T_\beta$$

$$\sum_{ab} W^{\alpha a,\beta b} = M^{\alpha\beta} - T_\alpha \delta_{\alpha\beta}$$

The set of equations for $W^{\alpha a,\beta b}$ is

$$W^{\alpha a,\beta b} = \tilde{M}_a^{\alpha\beta}\,\delta_{ab} + T_\alpha G_0 \sum_{(\gamma\neq\ \delta)\subset a}\ \sum_{a\supset\delta,\, d\neq a} M_a^{\alpha,\gamma} G_0\, W^{\delta d, b\beta}$$

with the free term

$$\tilde{M}_a^{\alpha\beta} = M_a^{\alpha\beta} - T_\alpha\,\delta_{ab}.$$

$W^{\alpha a,\beta b}$ is the sum of the almost connected graphs and

$$\tilde{W}^{\alpha a,\beta b} = W^{\alpha a,\beta b} - \tilde{M}_a^{\alpha\beta}$$

is the sum of the connected graphs.

$$\tilde{W}^{\alpha' a',\alpha a} = \tilde{M}_a^{\alpha'\alpha} G_0\, \tilde{M}_a^{\delta,\alpha}(1-\delta_{ab}) + \sum_{(\gamma\neq\ \delta)\subset a}\ \sum_{a\supset\delta,\, d\neq a'} M_a^{\alpha,\gamma} G_0\, \tilde{W}^{\delta d,\alpha a}$$

$$\tilde{W}^{\,\beta b,\alpha a} = \tilde{M}_b^{\beta\gamma}\, G_0\, \tilde{M}_a^{\delta,\alpha} + \sum_{(\gamma \neq\; \delta)\subset b} \; \sum_{d\neq b} M_b^{\beta,\gamma} G_0\, \tilde{W}^{\delta d,\alpha a}$$

We have 144 operators of the type $\tilde{W}^{\alpha' a';a\alpha}$ and 72 operators of the type $\tilde{W}^{\beta b;a\alpha}$.
In a system of identical particles the number of amplitudes is strongly reduced
The matrix elements of the operators $\tilde{W}^{\alpha' a;a\alpha}$ and $\tilde{W}^{\beta b\alpha}$ and be expressed in terms of 7 amplitudes $A_i\,(i = 1, \cdots, 7)$ and four amplitudes $B_i\,(i = 1, \cdots, 4)$.

Operators $\tilde{W}^{\alpha a;\, \alpha' a'}$	Examples	Matrix elements
$\alpha' = \alpha;\, a' = a$	$\tilde{W}^{12,123;\,12,123}$	A_1
$\alpha' \neq \alpha;\, a' = a$	$\tilde{W}^{12,123;\,31,123}$ $\tilde{W}^{12,123;\,23,123}$	A_2
$\alpha' \subset a;\, \alpha \subset a$	$\tilde{W}^{12,123;\,31,314}$, $\tilde{W}^{12,123;\,23,234}$	A_3
$\alpha' \subsetneq a;\, \alpha \subsetneq a'$	$\tilde{W}^{12,123;\,41,314}$, $\tilde{W}^{12,123;\,24,243}$	$A_4^\dagger$
$\alpha' \subsetneq a;\, \alpha \subsetneq a'$	$\tilde{W}^{12,123;\,43,314}$, $\tilde{W}^{12,123;\,34,342}$	$A_5^\dagger$
$\alpha' = \alpha,\, a' = a$	$\tilde{W}^{12,123;\,12,124}$	A_6
$(\alpha' \neq \alpha) \subset a$	$\tilde{W}^{12,123;\,14,142}$, $\tilde{W}^{12,123;\,24,241}$	A_7

† Amplitudes A_4 and A_5 differ in that the pairs α, α' have a common particle in the first case, and have no one in the second

We have the set of 11 equations relating A_i and B_i. We write down two typical equations of this set

$$\begin{aligned} A_1 \quad &= \quad 2\mathcal{M}^2 G_0 A_6 + 2[\mathcal{M}^1 + \mathcal{M}^2] G_0 A_7 + \\ & \eta^P \mathcal{M}^2 G_0 B_1 + 2\eta^P [\mathcal{M}^1 + \mathcal{M}^2] G_0 B_4, \end{aligned}$$

$$B_1 = \eta^P \, 2\mathcal{N}^{(2)} G_0 \tilde{\mathcal{M}}^{(1)} + 2\mathcal{N}^{(2)} G_0 (A_1 + A_6) + 2\mathcal{N}^{(1)} G_0 A_4$$

Equations of this type still contain too many amplitudes. A further simplification is possible, because the quantities with which we associate a physical meaning are not the amplitudes A_i, B_i themselves, but their linear combinations

$$\mathcal{A} = A_1 + 2A_2 + 2(A_3 + A_4 + A_5) + (A_6 + 2A_7)$$

$$\mathcal{B} = B_1 + B_2 + 2B_3 + 2B_4$$

$$\mathcal{A} = 2\mathcal{M} G_0 \tilde{\mathcal{M}} + 2\mathcal{M} G_0 \mathcal{A} + 4\eta_P \mathcal{M} G_0 \mathcal{B}$$

$$\mathcal{B} = 2\mathcal{N} G_0 \tilde{\mathcal{M}} + \mathcal{N} G_0 \mathcal{A}$$

$$\mathcal{M} = \mathcal{M}^{(1)} + 2\mathcal{M}^{(2)}, \qquad \mathcal{N} = \mathcal{N}^{(1)} + \eta^P \mathcal{N}^{(2)}$$

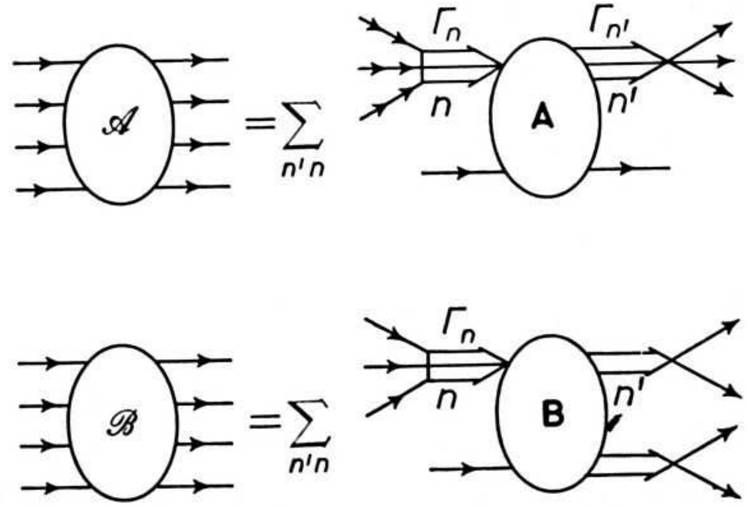


Figure 12: Pictorial definition of the amplitudes $\mathbb{A}_{nn'}$ and $\mathbb{B}_{nn'}$

$$F_{nt}(E, cos\theta) = -\frac{3\pi m}{4\,\gamma_\eta}\,\mathbb{A}_{11}(\mathbf{q}_f, \mathbf{q}_i; E + i0)$$

$$F_{nt\to dd}(E, cos\theta) = -\frac{\pi\,\sqrt{\mu_i\,\mu_f}}{\sqrt{\gamma_\eta\gamma_\zeta}}\,\left[\mathbb{B}_{11}(\mathbf{q}_f, \mathbf{q}_i; E + i0) + \mathbb{B}_{11}(-\mathbf{q}_f\,\mathbf{q}_i; E + i0)\right]$$

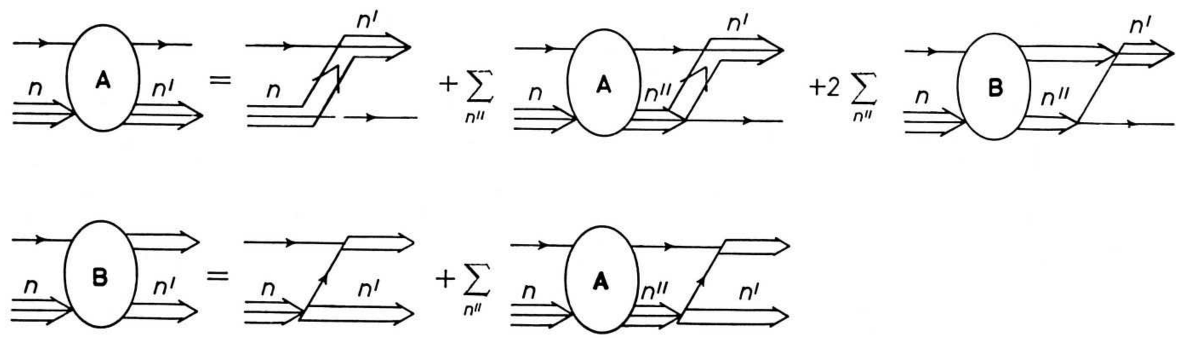


Figure 13: FY equations for $\mathbb{A}_{nn'}(3+1 \to 3+1)$ and $\mathbb{B}_{nn'}(3+1 \to 2+2)$

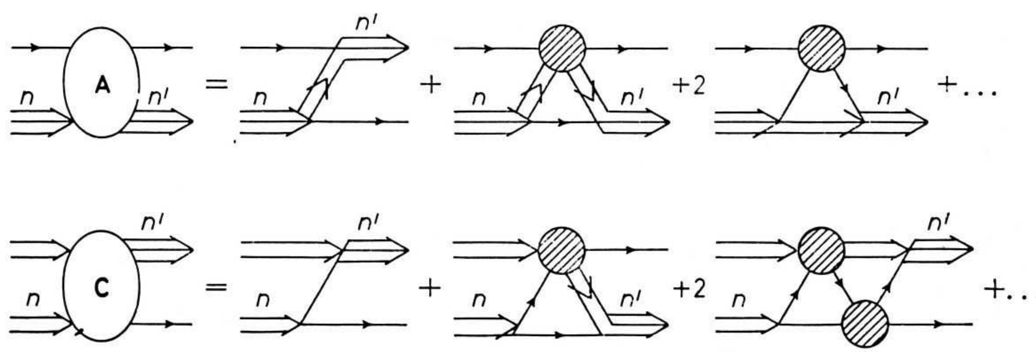


Figure 14: Iteration series for the amplitudes $\mathbb{A}$ and $\mathbb{C}$

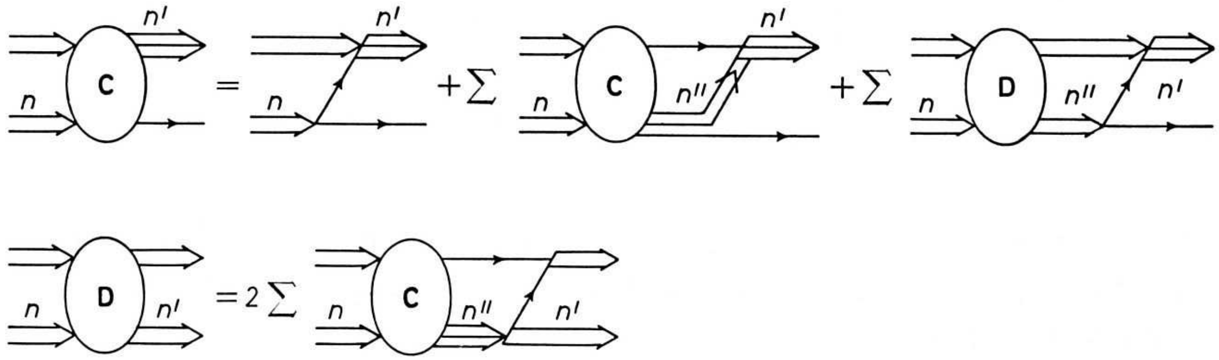


Figure 15: FY equations for $\mathbb{C}_{n\,n'}(2+2 \rightarrow 3+1)$ and $\mathbb{D}_{n\,n'}(2+2 \rightarrow 2+2)$

$$F_{dd\rightarrow nt}(E, cos\theta) = -\frac{\pi\sqrt{\mu_{nt}\mu_{dd}}}{\sqrt{\gamma_\eta\gamma_\zeta}} \left(\mathbb{C}_{11}(\mathbf{q}_f, \mathbf{q}_i; E+i0) + \mathbb{C}_{11}(\mathbf{q}_f, \mathbf{q}_i; E+i0)\right)$$

$$F_{dd\rightarrow dd}(E, cos\theta) = -\frac{\pi\mu_{dd}}{\gamma_\zeta} \left(\mathbb{D}_{11}(\mathbf{q}_f, \mathbf{q}_i; E+i0) + \mathbb{D}_{11}(-\mathbf{q}_f\ \mathbf{q}_i; E+i0)\right)$$

Four nucleon scattering. Applications

The theoretical description of the A=4 scattering states constitutes a serious challenge for the existing NN interaction models. The reason for that is not purely technical, but lies rather in the richness of the continuum spectrum itself. The 4N continuum spectrum exhibits a rich variety of resonances and thresholds sufficiently far from the zero energy region that cannot be determined by the low energy properties.

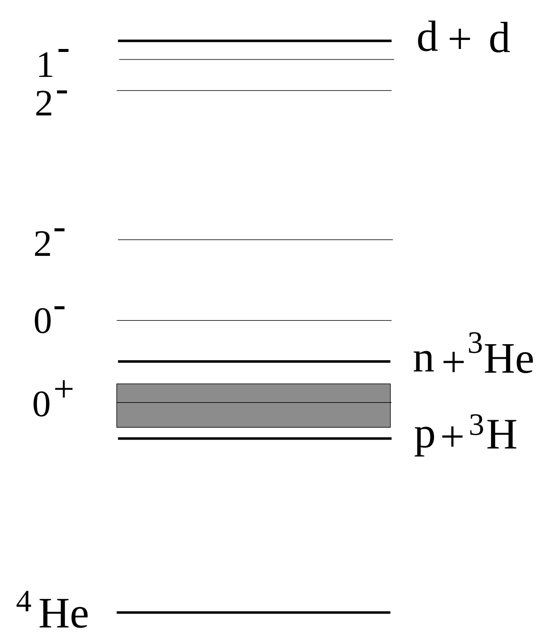


Figure 16: Isosinglet resonances and thresholds in 4N continuum spectrum

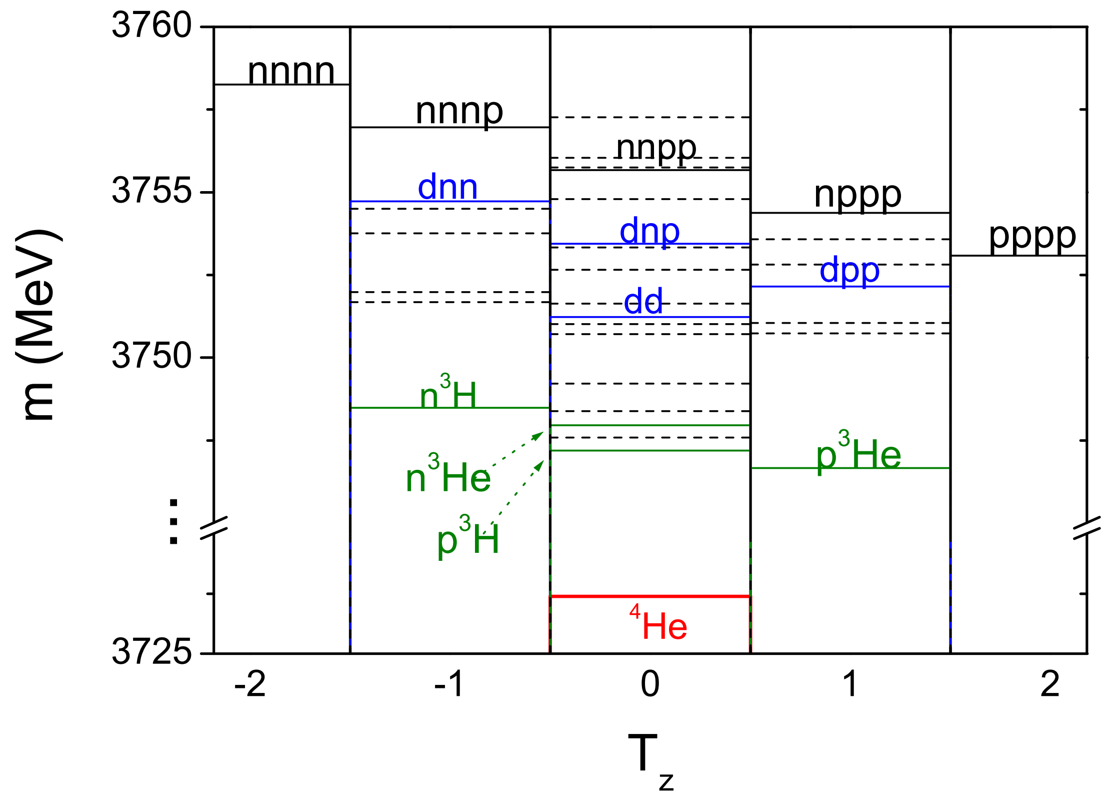


Figure 17: Spectra of 4N bound and resonate states

n-^{3}H, p-^{3}H, and p-^{3}He scattering lengths calculated using different interaction models

	MT I-III		AV 14		AV18+UIX	
	J$^\pi = 0^+$	J$^\pi = 1^+$	J$^\pi = 0^+$	J$^\pi = 1^+$	J$^\pi = 0^+$	J$^\pi = 1^+$
p^{3}He	11.5	9.20				
p^{3}H	-63.1	5.50	-13.9	5.77	-16.5	5.39
n^{3}H	4.10	3.63	4.28	3.81	4.04	3.60

$$\sigma_{nt}(0) \;=\; 177\,\mathrm{mb}\ (\mathrm{MT\ I-III}),\quad \sigma_{\mathrm{nt}}(0) \;=\; 194\,\mathrm{mb}\ (\mathrm{AV14}),$$
$$\sigma_{nt}(0) \;=\; 174\,\mathrm{mb}\ (\mathrm{AV18+UIX})$$

The comparison with the experimental cross section $\sigma(0) = 170 \pm 3$ mb from shows that the AV14 potentials fails in describing the zero energy cross section as it fails in reproducing B_3 and B_4, the three- and four-nucleon binding energies

Unlike the n-d case, the 4N scattering states call for three nucleon interaction from the very beginning

MT I-III potential was shown to be very successful in describing total and differential cross sections at the energies bellow nnd threshold. Situation is less obvious for realistic potentials, which require much larger partial wave basis to obtain converged results

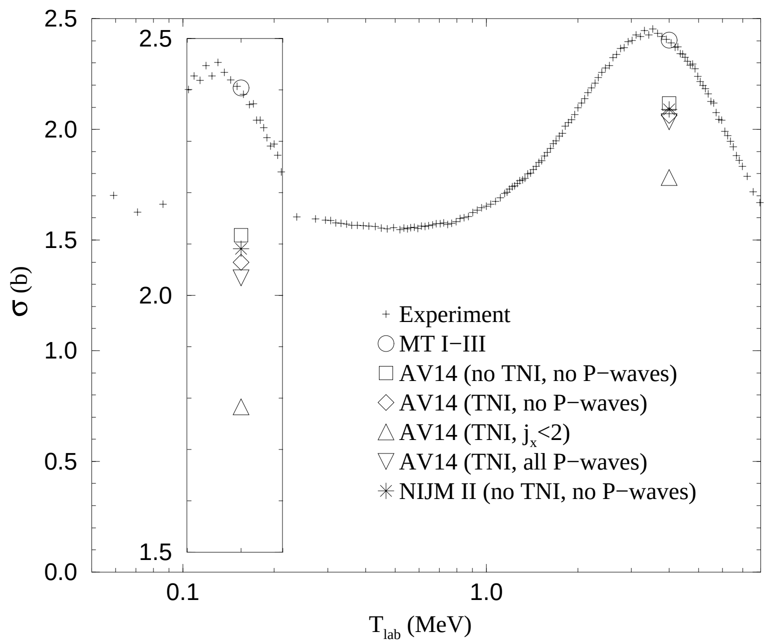


Figure 18: Predictions of $\sigma_{\mathrm{tot}}(\mathrm{n}^3\mathrm{H})$ for various interactions and experimental data

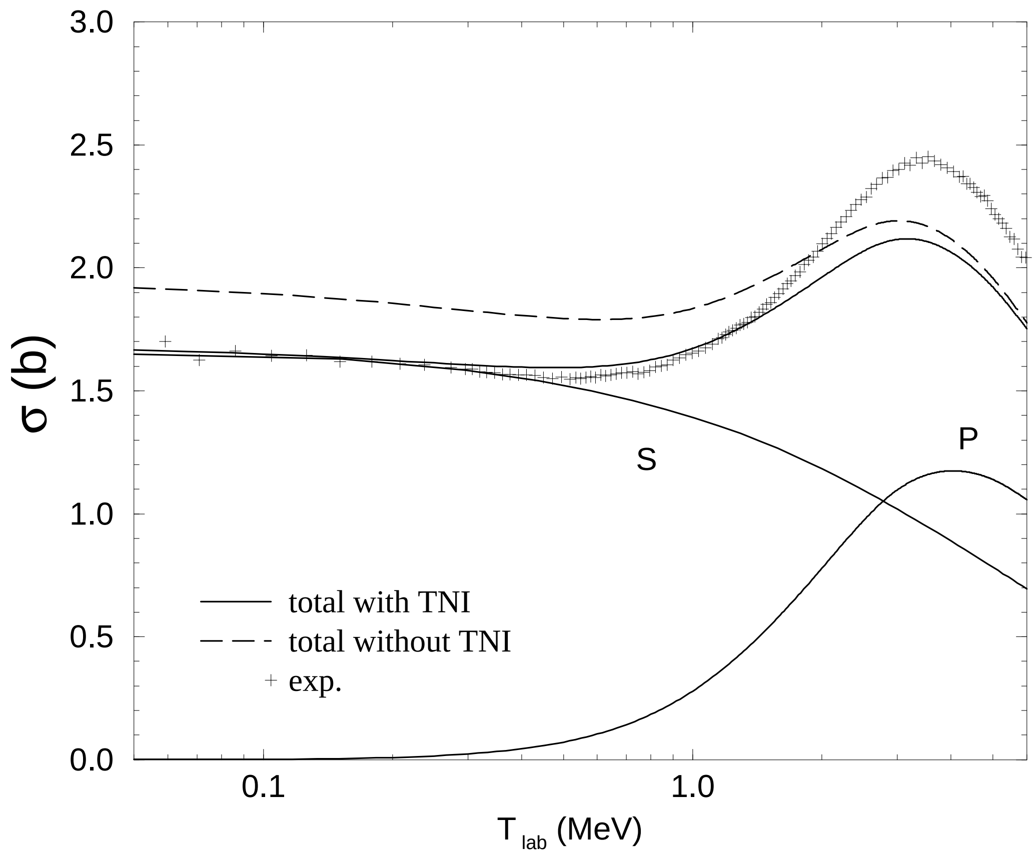


Figure 19: S and P waves in n^3H cross section

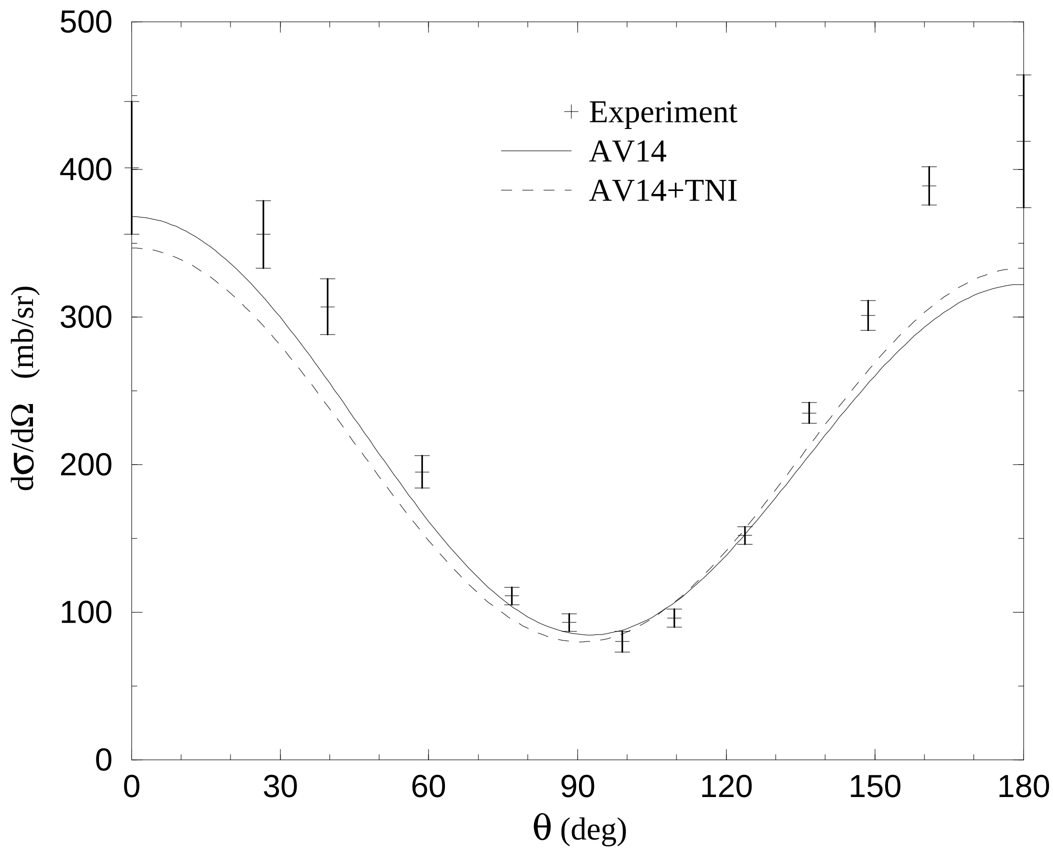


Figure 20: n^3H differential cross section

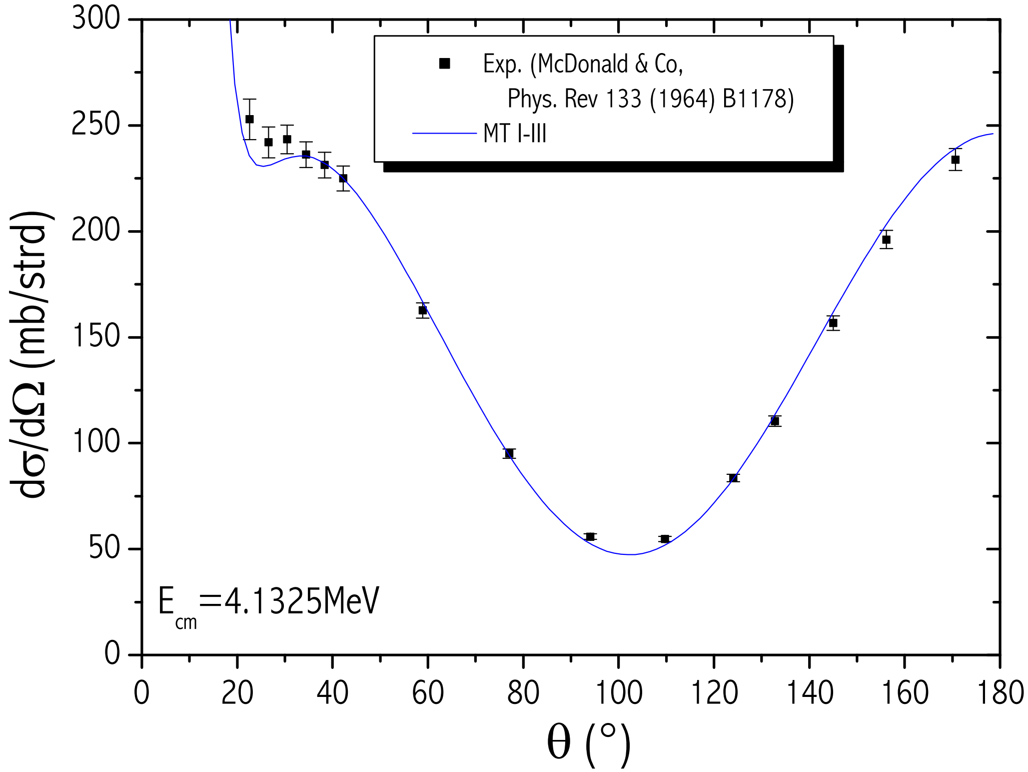


Figure 21: Differential cross sections versus centre-of-mass angle for p^{3}He at $E_{cm} = 4.1325$ MeV

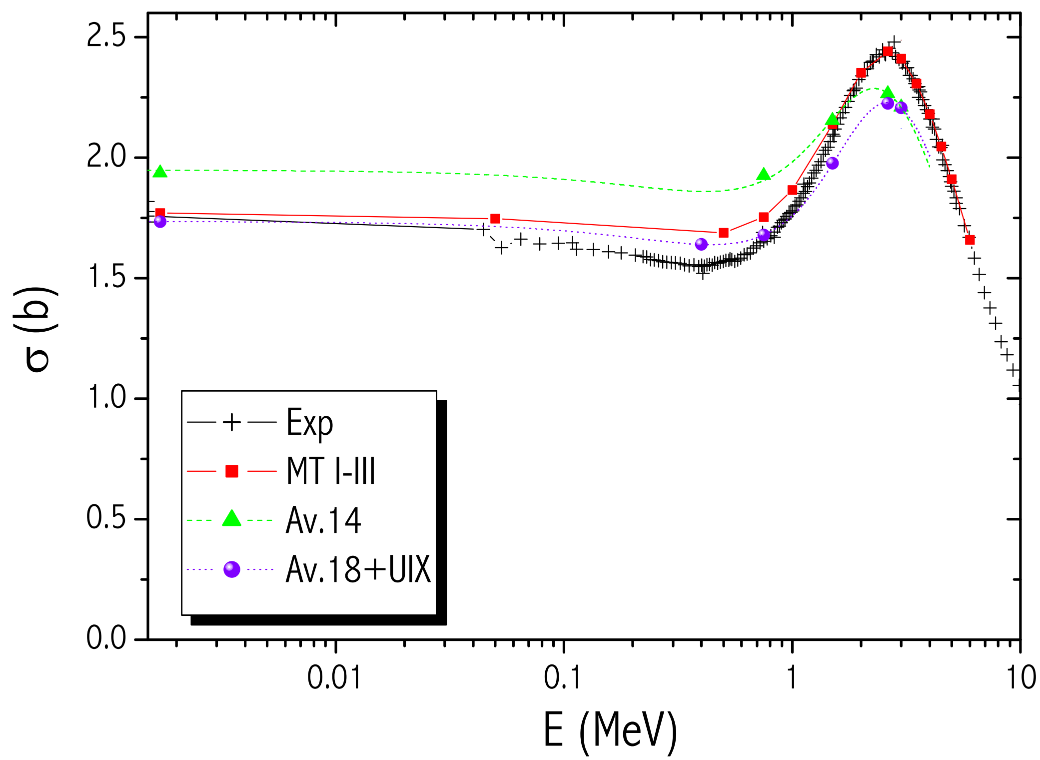


Figure 22: Theoretical n+^{3}H total cross sections compared with experimental data

Analysis of p$+{}^3$H scattering is complicated by the existence of the first 0^+ excitation of ^{4}He located at $E_R \approx 0.4\ MeV$ above p$+{}^3$H with the width $\Gamma \approx 0.5\ MeV$. By properly taking Coulomb interaction into account, thus separating n$+{}^3$He and p$+{}^3$H thresholds, it is possible to place the ^{4}He virtual state in between. However unlike in the other 4N systems, MT I-III predictions for a_s(p^{3}H) and $\sigma(E, \theta = 120^\circ)$ are in disagreement with data.

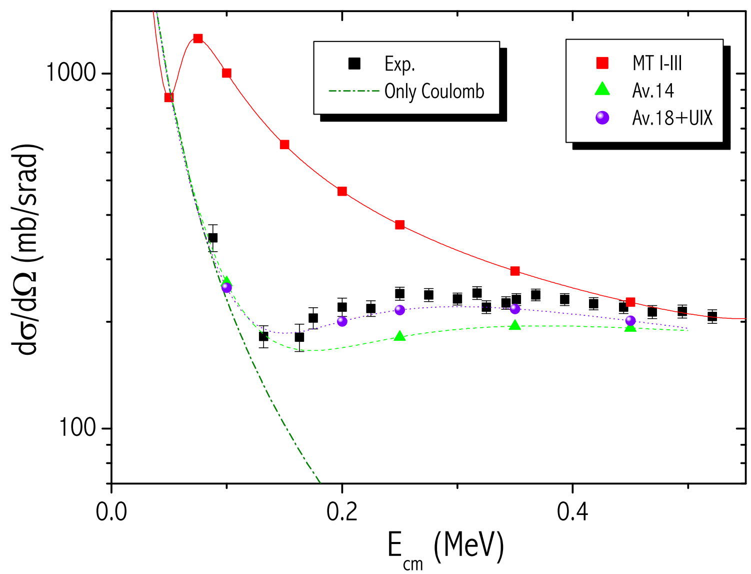


Figure 23: Energy dependence of p^{3}He elastic differential cross sections at 120°

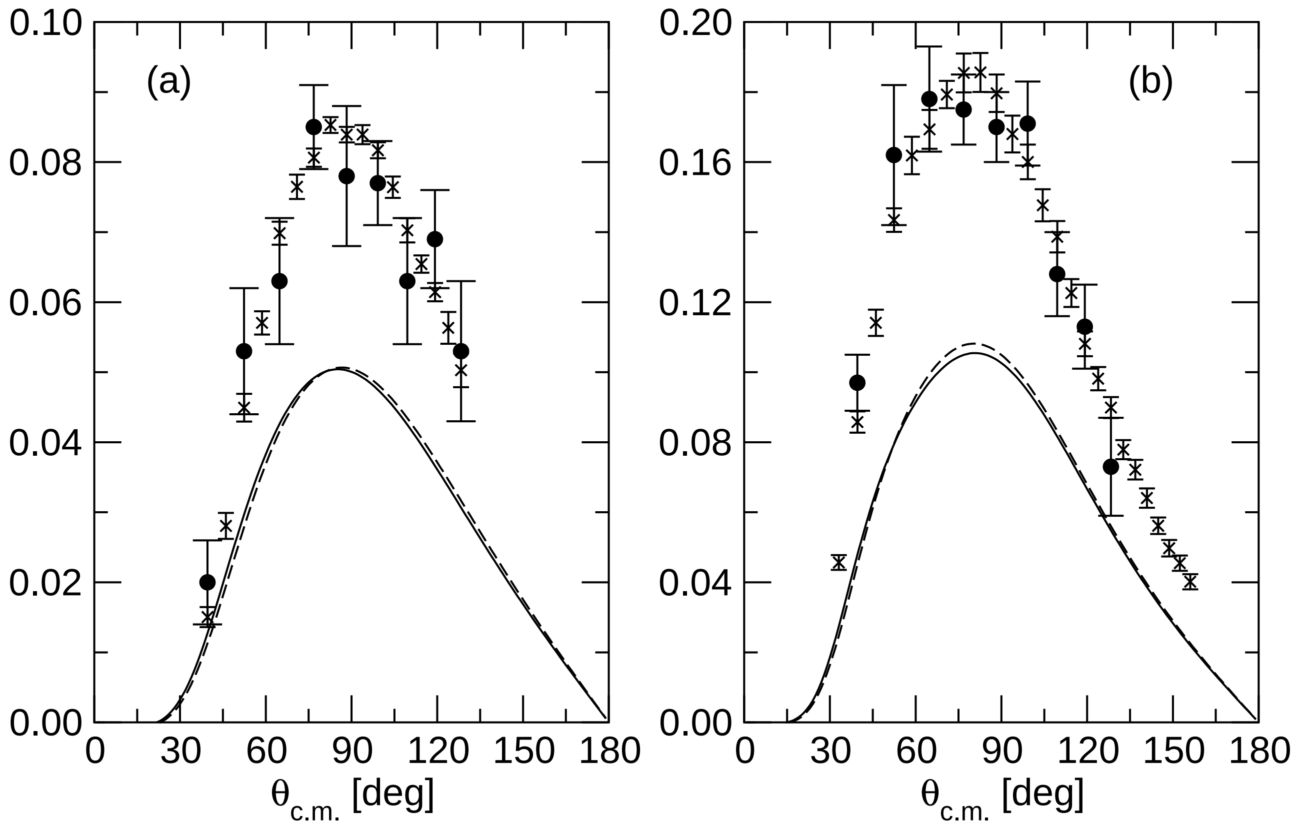


Figure 24: Vector analyzing power A_y versus θ_{CM} for $\vec{p}+{}^3$He reaction at $E_{cm} = 1.2$ MeV and 1.69 MeV respectively. The solid lines (dashed lines) correspond to calculations with AV18 alone, the dashed lines to AV18+ Urbana IX. Both calculations uncover large disagreements with experiment