Coulomb-Fourier transformation II.
Study of the ”Universal integral”

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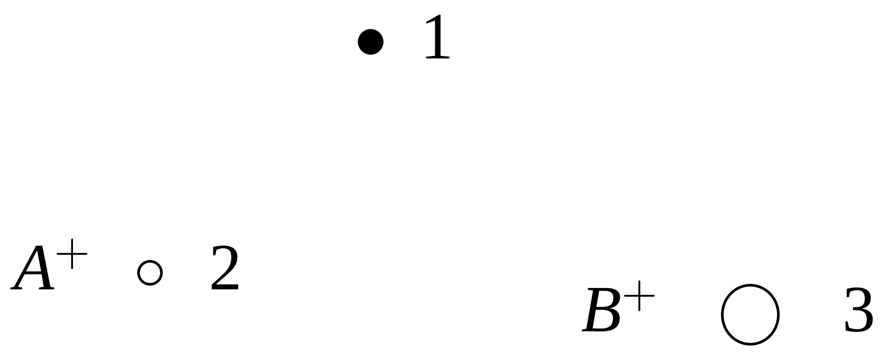
Based on the joint work with E.O.Alt and S.L.Yakovlev

E.O.Alt, S.B.Levin, S.L.Yakovlev, *Phys.Rev.C*, 69, 034002 (2004)

Description of the problem



Hamiltonian of the system:

$$\begin{aligned} H &= H_0 + V_1^{\mathrm{C}}(x_1) + \sum_{\nu=1}^{3} V_\nu(x_\nu) \\ &= -\Delta_{\mathbf{x}_1} - \Delta_{\mathbf{y}_1} + \frac{n_1}{x_1} + \sum_{\nu=1}^{3} V_\nu,\ n_1 > 0. \end{aligned} \tag{1}$$

$$\mathbf{x}_\alpha = c_{\alpha\beta}\mathbf{x}_\beta + s_{\alpha\beta}\mathbf{y}_\beta \tag{2}$$

$$\mathbf{y}_\alpha = -s_{\alpha\beta}\mathbf{x}_\beta + c_{\alpha\beta}\mathbf{y}_\beta. \tag{3}$$

The coefficients are given in terms of the particle masses $m_\nu,\ \nu = 1,2,3$, as

$$c_{\alpha\beta} = -\left[\frac{m_\alpha m_\beta}{(m_\alpha + m_\gamma)(m_\beta + m_\gamma)}\right]^{1/2}, \tag{4}$$

$$s_{\alpha\beta} = \varepsilon_{\beta\alpha}(1 - c_{\alpha\beta}^2)^{1/2}, \tag{5}$$

Schrödinger equation $H_0^{\mathrm{C}}\Psi = E\Psi$

$$H_0^{\mathrm{C}} = -\Delta_{\mathbf{y}_1} - \Delta_{\mathbf{x}_1} + \frac{n_1}{x_1},$$

admits an analitical solution

$$\Psi(\mathbf{X},\mathbf{P}) = \psi^{\mathrm{C}}_{\mathbf{k}_1}(\mathbf{x}_1)\psi^0_{\mathbf{p}_1}(\mathbf{y}_1),$$

where

$$\psi^{\mathrm{C}}_{\mathbf{k}_1}(\mathbf{x}_1) =$$
$$= (2\pi)^{-3/2}e^{i\mathbf{k}_1\cdot\mathbf{x}_1}e^{-\pi\gamma_1/2}\Gamma(1+i\gamma_1)\Phi(-i\gamma_1;1;ik_1\xi_1).$$

Here, $\xi_1 = x_1 - \mathbf{x}_1\cdot\hat{\mathbf{k}}_1$, $\gamma_1 = n_1/2k_1$ is the Sommerfield parameter, $\Phi(a;b;z)$ the confluent hypergeometric function, and $\Gamma(z)$ the Gamma function.

$$\psi^0_{\mathbf{p}_1}(\mathbf{y}_1) = (2\pi)^{-3/2}e^{i\mathbf{y}_1\cdot\mathbf{p}_1}.$$

It represents the normalized plane wave describing the free motion of particle 1 relative the center of mass of the pair $(2,3)$.

Because of the assumption $n_1 > 0$, the set $\{\psi^{\mathrm{C}}_{\mathbf{k}_1}, \psi^{0}_{\mathbf{p}_1}\}$ constitutes a basis in three-particle space which is complete, orthogonal, and normalized to δ-functions. With its help we construct a unitary three-body operator $\mathscr{F}_{\mathrm{C}}$ which acts as

$$\mathscr{F}_{\mathrm{C}} G(\mathbf{X}) = \int d\mathbf{P}\ \psi^{\mathrm{C}}_{\mathbf{k}_1}(\mathbf{x}_1)\psi^{0}_{\mathbf{p}_1}(\mathbf{y}_1)G(\mathbf{P}). \tag{6}$$

It effects a so-called Coulomb-Fourier ($\mathscr{C}\mathscr{F}$) transformation

The $\mathscr{C}\mathscr{F}$ transform of the Hamiltonian H separates into two terms,

$$\mathscr{H} := \mathscr{F}^{\dagger}_{\mathrm{C}} H \mathscr{F}_{\mathrm{C}} = \mathscr{T} + \sum_{\nu=1}^{3} \mathscr{V}_{\nu} \tag{7}$$

The kernel of the first term $\mathscr{T} := \mathscr{F}^{\dagger}_{\mathrm{C}}\{H_0 + V^{\mathrm{C}}_1\}\mathscr{F}_{\mathrm{C}}$ is easily seen to be

$$\mathscr{T}(\mathbf{P}', \mathbf{P}) = \mathbf{P}^2 \delta(\mathbf{P}' - \mathbf{P}), \tag{8}$$

where $\mathbf{P}^2 = \mathbf{k}_1^2 + \mathbf{p}_1^2$ $(= \mathbf{k}_\nu^2 + \mathbf{p}_\nu^2\ \forall\, \nu)$.

The $\mathscr{C}\mathscr{F}$-transformed short-range potentials

$$\mathscr{V}_\alpha \equiv \mathscr{F}_{\mathrm{C}}^\dagger \mathrm{V}_\alpha \mathscr{F}_{\mathrm{C}} \tag{9}$$

act as integral operators with kernels

$$\mathscr{V}_1(\mathbf{P}', \mathbf{P}) = \tilde{V}_1(\mathbf{k}_1', \mathbf{k}_1)\, \delta(\mathbf{p}_1' - \mathbf{p}_1),$$

$$\mathscr{V}_\alpha(\mathbf{P}', \mathbf{P}) = |s_{\alpha 1}|^{-3} \hat{V}_\alpha \left(\frac{\mathbf{p}_1' - \mathbf{p}_1}{s_{\alpha 1}} \right) \times$$
$$\times \mathscr{L}_\alpha^{\mathrm{C}}(\mathbf{P}', \mathbf{P}),\ \alpha = 2, 3.$$

Here,

$$\tilde{V}_1(\mathbf{k}_1', \mathbf{k}_1) = \int d\mathbf{x}_1 \, \psi^{\mathrm{C}*}_{\mathbf{k}_1'}(\mathbf{x}_1) V_1(\mathbf{x}_1) \, \psi^{\mathrm{C}}_{\mathbf{k}_1}(\mathbf{x}_1) \tag{10}$$

is the short-range potential of subsystem 1 in the “Coulomb representation”,

$$\hat{V}_\alpha(\mathbf{k}) = \int \frac{d\mathbf{x}_\alpha}{(2\pi)^3} \, e^{-i\mathbf{k}\cdot\mathbf{x}_\alpha} V_\alpha(\mathbf{x}_\alpha) \tag{11}$$

is the ordinary Fourier transform of $V_\alpha(\mathbf{x}_\alpha)$, and

$$\mathscr{L}^{\mathrm{C}}_\alpha(\mathbf{P}', \mathbf{P}) = \int d\mathbf{x}_1 \, e^{-i\tau_\alpha(\mathbf{p}_1 - \mathbf{p}_1')\cdot\mathbf{x}_1} \psi^{\mathrm{C}*}_{\mathbf{k}_1'}(\mathbf{x}_1) \, \psi^{\mathrm{C}}_{\mathbf{k}_1}(\mathbf{x}_1). \tag{12}$$

Moreover, τ_α is given in terms of the elements of the kinematic rotation matrix as $\tau_\alpha := c_{\alpha 1}/s_{\alpha 1}$.

Representation of the kernel $\mathscr{V}_\alpha$ in associated configuration space

To present the kernel $\mathscr{V}_\alpha$ in associated configuration space we need to calculate an inverse Fourier transform

$$\tilde{\mathscr{V}}_\alpha(\mathbf{X},\mathbf{X}') = \mathscr{F}_O \mathscr{V}_\alpha \mathscr{F}_O^\dagger =$$

$$= (2\pi)^{-6}\int_{\mathbf{R}^3} d\mathbf{k}_\alpha \int_{\mathbf{R}^3} d\mathbf{k}'_\alpha \int_{\mathbf{R}^3} d\mathbf{p}_\alpha \int_{\mathbf{R}^3} d\mathbf{p}'_\alpha \mathscr{V}_\alpha(\mathbf{P}',\mathbf{P})\times$$

$$\times e^{-i<\mathbf{k}_\alpha,\mathbf{x}_\alpha>} e^{i<\mathbf{k}'_\alpha,\mathbf{x}'_\alpha>} e^{-i<\mathbf{p}_\alpha,\mathbf{y}_\alpha>} e^{i<\mathbf{p}'_\alpha,\mathbf{y}'_\alpha>},$$

where

$$\mathscr{V}_\alpha(\mathbf{P}',\mathbf{P}) \simeq \hat{V}_\alpha(\mathbf{k}_\alpha - \mathbf{k}'_\alpha)\mathscr{A}(\mathbf{k}_1,\mathbf{k}'_1)\delta(\mathbf{p}_\alpha - \mathbf{p}'_\alpha), \quad \alpha = 2,3.$$

The coefficient $\mathscr{A}(\mathbf{k}_1, \mathbf{k}_1')$ described as

$$\mathscr{A}(\mathbf{k}_1, \mathbf{k}_1') = (2k_1)^{i\gamma_1}(2k_1')^{-i\gamma_1'}\Gamma(1+i\gamma_1)\Gamma(1-i\gamma_1')\times$$
$$\times\frac{\sinh\left\{\frac{\pi}{2}(\gamma_1-\gamma_1')\right\}}{\frac{\pi}{2}(\gamma_1-\gamma_1')}{}_2F_1\left(-i\gamma_1, i\gamma_1'; 1; \frac{1+ < \hat{\mathbf{k}}_1, \hat{\mathbf{k}}_1' >}{2}\right).$$

We know also

$$\mathbf{k}_1 = c_{\alpha 1}\mathbf{k}_\alpha - s_{\alpha 1}\mathbf{p}_\alpha, \qquad \gamma_1 = \frac{n}{2k_1},$$
$$\mathbf{k}_1' = c_{\alpha 1}\mathbf{k}_\alpha' - s_{\alpha 1}\mathbf{p}_\alpha, \qquad \gamma_1' = \frac{n}{2k_1'},$$

After integration over $d\mathbf{p}'_\alpha$, instead of the set of three independent variables

$$\{\mathbf{k}_\alpha, \mathbf{k}'_\alpha, \mathbf{p}_\alpha\}$$

we introduce the new set of three independent variables

$$\{\mathbf{q}_\alpha, \mathbf{q}'_\alpha, \mathbf{p}_\alpha\},$$

where

$$\mathbf{q}_\alpha = \mathbf{k}_\alpha - \frac{s_{\alpha 1}}{c_{\alpha 1}}\mathbf{p}_\alpha, \qquad \mathbf{q}'_\alpha = \mathbf{k}'_\alpha - \frac{s_{\alpha 1}}{c_{\alpha 1}}\mathbf{p}_\alpha.$$

In new variables

$$\tilde{\mathscr{V}}_\alpha(\mathbf{X},\mathbf{X}') = \mathscr{F}_O \mathscr{V}_\alpha \mathscr{F}_O^\dagger =$$

$$= (2\pi)^{-6} \int_{\mathbf{R}^3} d\mathbf{q}_\alpha \int_{\mathbf{R}^3} d\mathbf{q}'_\alpha \int_{\mathbf{R}^3} d\mathbf{p}_\alpha \hat{V}_\alpha\left(\mathbf{q}_\alpha - \mathbf{q}'_\alpha\right) \mathscr{A}\left(c_{\alpha 1}\mathbf{q}_\alpha, c_{\alpha 1}\mathbf{q}'_\alpha\right) \times$$

$$\times e^{-i<\mathbf{q}_\alpha,\mathbf{x}_\alpha>} e^{i<\mathbf{q}'_\alpha,\mathbf{x}'_\alpha>} e^{-i\frac{1}{c_{\alpha 1}}<\mathbf{p}_\alpha,\mathbf{y}_1-\mathbf{y}'_1>} =$$

$$= \left(\frac{|c_{\alpha 1}|}{2\pi}\right)^3 \delta(\mathbf{y}_1 - \mathbf{y}'_1) \times$$

$$\times \int_{\mathbf{R}^3} d\mathbf{q}_\alpha \int_{\mathbf{R}^3} d\mathbf{q}'_\alpha \hat{V}_\alpha\left(\mathbf{q}_\alpha - \mathbf{q}'_\alpha\right) \mathscr{A}\left(c_{\alpha 1}\mathbf{q}_\alpha, c_{\alpha 1}\mathbf{q}'_\alpha\right) \times$$

$$\times e^{-i<\mathbf{q}_\alpha,\mathbf{x}_\alpha>} e^{i<\mathbf{q}'_\alpha,\mathbf{x}'_\alpha>}.$$

In new variables

$$\mathbf{p} = \frac{1}{\sqrt{2}}(\mathbf{q}_\alpha - \mathbf{q}'_\alpha), \quad \mathbf{k} = \frac{1}{\sqrt{2}}(\mathbf{q}_\alpha + \mathbf{q}'_\alpha)$$

we use the fact, that potential depends on the variable $\mathbf{p}$ only. We use also that with special properties:

$$\mathscr{A}(\mathbf{p},\mathbf{k}) \underset{k\to\infty,\ p\le R<\infty}{\longrightarrow} 1,$$

$$\mathscr{A}(\mathbf{p},\mathbf{k}) \underset{p\to 0}{\longrightarrow} 1.$$

Finally

$$\tilde{\mathscr{V}}_\alpha(\mathbf{X},\mathbf{X}') = \mathscr{F}\mathscr{V}_\alpha\mathscr{F}^\dagger =$$

$$= \left(\frac{|c_{\alpha 1}|}{2\pi}\right)^3 \delta(\mathbf{y}_1 - \mathbf{y}'_1) \int_{\mathbf{R}^3} d\mathbf{p}\hat{V}_\alpha(\mathbf{p}) \int_{\mathbf{R}^3} d\mathbf{k}[\mathscr{A}(\mathbf{p},\mathbf{k}) - 1]\times$$

$$\times e^{-\frac{i}{\sqrt{2}}<\mathbf{k}+\mathbf{p},\mathbf{x}_\alpha>} e^{\frac{i}{\sqrt{2}}<\mathbf{k}-\mathbf{p},\mathbf{x}'_\alpha>} +$$

$$+|c_{\alpha 1}|^3 \delta(\mathbf{y}_1 - \mathbf{y}'_1)\delta(\mathbf{x}_\alpha - \mathbf{x}'_\alpha)V(x_\alpha).$$

Since $\mathbf{y}_1 = c_{\alpha 1}\mathbf{y}_\alpha + s_{\alpha 1}\mathbf{x}_\alpha$, the second asymptotic term appears as

$$\delta(\mathbf{y}_\alpha - \mathbf{y}'_\alpha)\delta(\mathbf{x}_\alpha - \mathbf{x}'_\alpha)V_\alpha(x_\alpha)$$

and describes the asymptotic behavior of transformed pair potential in pair $\alpha = 2,3$.
The first term reflects the complicated structure of transformed initial pair potential at small distances.

Conclusion

We proved, that by Coulomb Fourier transformation we eliminated the long-range Coulomb interaction from the Hamiltonian and preserved the asymptotic structure of other pair potentials.

Derivation of the "Universal integral"

We calculate first an expression for the kernel of the potential in pairs, which contain a neutral particle.

$$\mathcal{V}_\alpha(\mathbf{P}',\mathbf{P}) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} d\mathbf{x}_1 \int_{\mathbf{R}^3} d\mathbf{x}'_1 \int_{\mathbf{R}^3} d\mathbf{y}_1 \int_{\mathbf{R}^3} d\mathbf{y}'_1 \psi_c(\mathbf{x}_1,\mathbf{k}_1)\times$$

$$\times\psi_c^*(\mathbf{x}'_1,\mathbf{k}'_1)e^{i<\mathbf{y}_1,\mathbf{p}_1>}e^{-i<\mathbf{y}'_1,\mathbf{p}'_1>}\delta(\mathbf{y}_\alpha-\mathbf{y}'_\alpha)\delta(\mathbf{x}_\alpha-\mathbf{x}'_\alpha)V(\mathbf{x}_\alpha).$$

Instead one set of independent variables $\{\mathbf{x}_1,\mathbf{y}_1\}$ we choose another one $\{\mathbf{x}_1,\mathbf{x}_\alpha\}$, where

$$\mathbf{x}_\alpha = c_{\alpha 1}\mathbf{x}_1 + s_{\alpha 1}\mathbf{y}_1.$$

The Jacobian of one-dimensional transformation

$$x_\alpha = c_{\alpha 1} x_1 + s_{\alpha 1} y_1$$

is

$$J^{(N=1)} = \begin{vmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial y_1} \\ \frac{\partial x_\alpha}{\partial x_1} & \frac{\partial x_\alpha}{\partial y_1} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ c_{\alpha 1} & s_{\alpha 1} \end{vmatrix} = s_{\alpha 1}.$$

Since it will be valid in each dimension, the three-dimensional Jacobian is the power three of one-dimensional one

$$J^{(N=3)} = s_{\alpha 1}^3.$$

The same will be valid for the primed coordinates $\{\mathbf{x}'_1, \mathbf{x}'_\alpha\}$.

We also take into account that

$$\mathbf{y}_\alpha = -\frac{1}{s_{\alpha 1}}\mathbf{x}_1 + \frac{c_{\alpha 1}}{s_{\alpha 1}}\mathbf{x}_\alpha,$$

and consequently

$$\mathbf{y}_\alpha - \mathbf{y}'_\alpha = -\frac{1}{s_{\alpha 1}}(\mathbf{x}_1 - \mathbf{x}'_1) + \frac{c_{\alpha 1}}{s_{\alpha 1}}(\mathbf{x}_\alpha - \mathbf{x}'_\alpha).$$

It means, that

$$\delta(\mathbf{x}_\alpha - \mathbf{x}'_\alpha)\delta(\mathbf{y}_\alpha - \mathbf{y}'_\alpha) = \delta(\mathbf{x}_\alpha - \mathbf{x}'_\alpha)\delta\left(\frac{\mathbf{x}_1 - \mathbf{x}'_1}{s_{\alpha 1}}\right) =$$

$$= |s_{\alpha 1}|^3 \delta(\mathbf{x}_\alpha - \mathbf{x}'_\alpha)\delta\left(\mathbf{x}_1 - \mathbf{x}'_1\right).$$

Integrating over the primed coordinates, we derive

$$\mathscr{V}_\alpha(\mathbf{P}', \mathbf{P}) = \frac{1}{|s_{\alpha 1}|^3 (2\pi)^3} \int_{\mathbf{R}^3} d\mathbf{x}_1 \int_{\mathbf{R}^3} d\mathbf{x}_\alpha \psi_c(\mathbf{x}_1, \mathbf{k}_1) \times$$

$$\times \psi_c^*(\mathbf{x}_1, \mathbf{k}'_1) e^{i\frac{1}{s_{\alpha 1}} < \mathbf{x}_\alpha, \mathbf{p}_1 - \mathbf{p}'_1 >} e^{-i\frac{c_{\alpha 1}}{s_{\alpha 1}} < \mathbf{x}_1, \mathbf{p}_1 - \mathbf{p}'_1 >} V(\mathbf{x}_\alpha).$$

Integrating over $d\mathbf{x}_\alpha$ we use the separation of variables and the definition of the Fourier transform

$$\hat{V}_\alpha\left(\frac{\mathbf{p}_1' - \mathbf{p}_1}{s_{\alpha 1}}\right) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} d\mathbf{x}_\alpha e^{i\frac{1}{s_{\alpha 1}}<\mathbf{x}_\alpha,\mathbf{p}_1-\mathbf{p}_1'>} V(\mathbf{x}_\alpha).$$

Therefor we finally derive an expression

$$\mathscr{V}_\alpha(\mathbf{P}',\mathbf{P}) = |s_{\alpha 1}|^{-3}\hat{V}_\alpha\left(\frac{\mathbf{p}_1' - \mathbf{p}_1}{s_{\alpha 1}}\right) \times$$

$$\times \int_{\mathbf{R}^3} d\mathbf{x}_1 \psi_c(\mathbf{x}_1,\mathbf{k}_1)\psi_c^*(\mathbf{x}_1,\mathbf{k}_1')e^{-i\frac{c_{\alpha 1}}{s_{\alpha 1}}<\mathbf{x}_1,\mathbf{p}_1-\mathbf{p}_1'>}.$$

Consider the limit case $n = Z_1 Z_2 e^2 \sqrt{2\mu} \to 0$**.**
Then

$$\psi_c(\mathbf{x}_1, \mathbf{k}_1) \underset{n\to 0}{\longrightarrow} (2\pi)^{-3/2} e^{i<\mathbf{x}_1, \mathbf{k}_1>},$$

which consequently means

$$\mathscr{V}_\alpha(\mathbf{P}', \mathbf{P}) \underset{n\to 0}{\longrightarrow} \delta(\mathbf{p}_\alpha - \mathbf{p}'_\alpha) \hat{V}_\alpha(\mathbf{k}_\alpha - \mathbf{k}'_\alpha).$$

Coming back to the main expression, we finally proved, that

$$\mathscr{V}_\alpha(\mathbf{P}', \mathbf{P}) = |s_{\alpha 1}|^{-3} \hat{V}_\alpha \left(\frac{\mathbf{p}'_1 - \mathbf{p}_1}{s_{\alpha 1}} \right) \times$$

$$\times \mathscr{L}^C_\alpha(\mathbf{P}', \mathbf{P}), \ \alpha = 2, 3.$$

Evaluation of analytical expression for the "Universal integral" outside of vicinities of singularities

Has been studied in atomic physics by H.Bethe, and A.Nordsieck in 1950-th and later by M.Gravielle and J.Miraglia

A. Nordsieck, Phys.Rev., v.93, 785, (1954)
M. S. Gravielle and J. E. Miraglia, Comp. Phys. Com. **69**, *59, (1992).*

We calculate now the integral

$$\mathscr{L} = \int_{\mathbf{R}^3} d\mathbf{r} e^{-\lambda r + i\mathbf{q}\cdot\mathbf{r}} \Phi(-i\gamma, 1, i(kr - \mathbf{k}\cdot\mathbf{r}))\Phi(i\gamma', 1, -i(k'r - \mathbf{k}'\cdot\mathbf{r})), \quad (13)$$

where λ is a regularization parameter. The limit procedure

$$\lambda \to 0$$

will define $\mathscr{L}$ as a distribution.

We first consider the integral

$$\mathscr{L}_1 = \int_{\mathbf{R}^3} \frac{d\mathbf{r}}{r} e^{-\lambda r + i\mathbf{q}\cdot\mathbf{r}} \Phi(-i\gamma, 1, i(kr - \mathbf{k}\cdot\mathbf{r}))\Phi(i\gamma', 1, -i(k'r - \mathbf{k}'\cdot\mathbf{r})) \quad (14)$$

which is of the Nordsieck type.

From $\mathscr{L}_1$ the desired integral follows directly as

$$\mathscr{L} = -\frac{\partial}{\partial\lambda}\mathscr{L}_1. \quad (15)$$

To evaluate $\mathscr{L}_1$ we employ the integral representation for the confluent hypergeometric function $\Phi(a,1,z)$,

$$\Phi(a,1,z) = -\frac{1}{2\pi i}\oint_{C_1} e^{tz}(-t)^{a-1}(1-t)^{-a}dt, \tag{16}$$

where the contour C_1 starts at the point $t=1$ and encircles the point $t=0$ once in the positive sense. With its help we rewrite (14) as

$$\begin{aligned}\mathscr{L}_1 \;=\; & -\frac{1}{4\pi^2}\oint_{C_1} dt_1 \oint_{C_2} dt_2 \left(\frac{t_1-1}{t_1}\right)^{i\gamma}\left(\frac{t_2}{t_2-1}\right)^{i\gamma'}\frac{1}{t_1 t_2} \\ & \times \int_{\mathbf{R}^3}\frac{d\mathbf{r}}{r} e^{-\lambda r + i(kt_1 - k't_2)r + i(\mathbf{q}+t_2\mathbf{k}'-t_1\mathbf{k})\cdot\mathbf{r}}.\end{aligned} \tag{17}$$

We consider first an arbitrary non-collinear vectors $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{k}'$

$$[\mathbf{k}\times\mathbf{k}']\neq 0,\quad [\mathbf{q}\times\mathbf{k}']\neq 0,\quad [\mathbf{q}\times\mathbf{k}]\neq 0, \tag{18}$$

with an additional condition

$$\mathbf{q}\notin\Pi(\mathbf{k},\mathbf{k}'), \tag{19}$$

where $\Pi(\mathbf{k},\mathbf{k}')$ is a plane, constructed on vectors $\mathbf{k}$ and $\mathbf{k}'$. In order that change of the order of integration be permitted the convergence condition

$$\lambda+2k\Im(t_1)-2k'\Im(t_2)>0,\quad \forall t_1\in C_1,\quad \forall t_2\in C_2, \tag{20}$$

must be satisfied. Here $\Im$ means imaginary part of the complex expression. The same condition must hold for any point inside the contours.

After performing the integration over $\mathbf{r}$ we obtain

$$\begin{aligned}\mathscr{L}_1 \;=\; &-\frac{1}{\pi}\oint_{C_1}dt_1\oint_{C_2}dt_2\left(\frac{t_1-1}{t_1}\right)^{i\gamma}\left(\frac{t_2}{t_2-1}\right)^{i\gamma'}\frac{1}{t_1t_2}\\ &\times\frac{1}{[(\lambda-i(kt_1-k't_2))^2+(\mathbf{q}-t_1\mathbf{k}+t_2\mathbf{k}')^2]}.\end{aligned} \tag{21}$$

Extracting the additional pole in the variable t_1 which comes from the term in square brackets, this is equivalent to

$$\begin{aligned}\mathscr{L}_1 &= \frac{1}{2\pi}\oint_{C_1} dt_1 \oint_{C_2} dt_2 \left(\frac{t_1-1}{t_1}\right)^{i\gamma}\left(\frac{t_2}{t_2-1}\right)^{i\gamma'}\frac{1}{t_1t_2} \\ &\times\frac{1}{[U-t_2V]}\frac{1}{[t_1-t_1^{(0)}]},\end{aligned} \tag{22}$$

with

$$t_1^{(0)} = \frac{T+2t_2U'}{2(U-t_2V)}, \tag{23}$$

Here, we have introduced the abbreviations

$$U = \mathbf{q}\cdot\mathbf{k}+i\lambda k,\; U' = \mathbf{q}\cdot\mathbf{k}'+i\lambda k', \tag{24}$$

$$V = kk'-\mathbf{k}\cdot\mathbf{k}',\; T = q^2+\lambda^2. \tag{25}$$

The location of the pole $t_1 = t_1^{(0)}$ can be found by considering

$$S_0 := \lambda+2k\Im(t_1^{(0)})-2k'\Im(t_2). \tag{26}$$

Taking in account that the imaginary part of the variable t_2 can be made arbitrarily small, this reduces to investigating

$$S_0 = \lambda + 2k\Im(t_1^{(0)}). \tag{27}$$

A simple calculation gives

$$S_0 = -\frac{\lambda}{D}\{q^2k^2 - <\mathbf{q}\cdot\mathbf{k}>^2 + t_2^2[k^2k'^2 - <\mathbf{k}\cdot\mathbf{k}'>^2] + \tag{28}$$

$$+2t_2[k^2<\mathbf{q}\cdot\mathbf{k}'> - <\mathbf{q}\cdot\mathbf{k}><\mathbf{k}\cdot\mathbf{k}'>]\} =$$

$$= -\frac{\lambda}{D}|[[\mathbf{q}\times\mathbf{k}]\times\hat{\mathbf{q}}] + t_2[[\mathbf{q}\times\mathbf{k}']\times\hat{\mathbf{q}}]|^2 < 0, \tag{29}$$

for arbitrary $t_2 \in [0,1]$ in accordence with assumptions (18)-(19). We introduced here the new notation D:

$$D = [\mathbf{q}\cdot\mathbf{k} - t_2(kk' - \mathbf{k}\cdot\mathbf{k}')]^2 + \lambda^2k^2. \tag{30}$$

Since the expression (26) is a smooth function of both $\Im(t_2)$ and $\Re(t_2)$, one can allways find in the complex plane t_2 a small vicinity U_ε of the real axis segment $t_2 \in [0,1]$, where nonequality (29) still will be valid. We construct the contours C_1 and C_2 in equation (17) in this small vicinity U_ε.

Comparing nonequality (29) with (20) we conclude that the point $t_1^{(0)}$ lies outside the contour C_1, and that it gives rise the only singularity there. Hence, we can apply Cauchy's theorem to the region bounded by C_1 and a circle with radius $R \to \infty$ to perform the integration over t_1, and obtain

$$\mathscr{L}_1 = -\frac{2i}{T}\left(1-\frac{2U}{T}\right)^{i\gamma}\left(1+\frac{2U'}{T}\right)^{-i\gamma'} \times \tag{31}$$
$$\times \oint_{C_4}^{(0+,1+)} d\tau (1-x_0\tau)^{-i\gamma'}\tau^{-i\gamma-1}(\tau-1)^{i\gamma}.$$

Comparing expression (31) with the definition of hypergeometric function,

$${}_2F_1(\zeta,\eta,1;x) = -\frac{i}{2\pi}\oint^{(0+,1+)} dt t^{\eta-1}(t-1)^{-\eta}(1-tx)^{-\zeta}, \tag{32}$$

we deduce

$$\mathscr{L}_1 = \frac{4\pi}{q^2+\lambda^2} A_1^{i\gamma} A_2^{-i\gamma'} {}_2F_1(i\gamma', -i\gamma, 1; x_0), \tag{33}$$

with

$$x_0 = 1 + \frac{|\mathbf{q}+\mathbf{k}'-\mathbf{k}|^2 - (k'-k-i\lambda)^2}{(q^2+\lambda^2)A_1A_2}, \tag{34}$$

where

$$A_1 = -\frac{|\mathbf{q}-\mathbf{k}|^2 - (k+i\lambda)^2}{q^2+\lambda^2}, \qquad A_2 = \frac{|\mathbf{q}+\mathbf{k}'|^2 - (k'-i\lambda)^2}{q^2+\lambda^2}. \tag{35}$$

Note, that derived expression (33) is an analytical function of vectors $\mathbf{q}$, $\mathbf{k}$ and $\mathbf{k}'$.

Now one can come back to conditions (18)-(19). Consider first the situation when vector q has an arbitrary small projection, which is orthogonal to the plane Π, constructed on vectors $\mathbf{k}$ and $\mathbf{k}'$. Because of the analyticity of function $\mathscr{L}_1$ (33) on variable $\mathbf{q}$, the expression (33) should be also valid for the case $\mathbf{q} \in \Pi$, when the condition (19) is violated.

In the same way one can prove, that analytical expression (33) is also valid, when the condition (18) is violated.

Finally we proved, that expression (33) is valid for arbitrary vectors **q**, **k** and $\mathbf{k}'$.

Finally we derive an expression for $\mathscr{L}$ as the following:

$$\mathscr{L} = \frac{4\pi}{(q^2+\lambda^2)^2} A_1^{i\gamma} A_2^{-i\gamma'} \left[J_{21} \mathscr{F}(x_0) + \frac{\gamma\gamma'}{A_1 A_2} J_{22} \mathscr{F}^+(x_0) \right]. \tag{36}$$

We introduced the short-hand notations

$$\mathscr{F}(x_0) := {}_2F_1(i\gamma', -i\gamma, 1; x_0), \quad \mathscr{F}^+(x_0) := {}_2F_1(1+i\gamma', 1-i\gamma, 2; x_0). \tag{37}$$

Clearly, this limiting procedure $\lambda \to 0$ will define $\mathscr{L}^C(\mathbf{P}', \mathbf{P})$ only in the sense of distributions.

We need to separate the most singular part of the distribution $\mathscr{L}^C(\mathbf{P}', \mathbf{P})$, because it will clarify the asymptotic behavior of the Coulomb Fourier transformed potential in configuration space.

The complicated non-uniform angular dependence in initial parabolic coordinates do not allowed to separate the most singular term immediately from full expression (36).

We present two different ways to avoid the problems, mentioned above.

Derivation of the most singular part of $\mathscr{L}_c(\mathbf{P}', \mathbf{P})$.
1. Weak asymptotics method

Let us start from the defining expression which is more carefully written as

$$\mathscr{L}_\alpha^{\mathrm{C}}(\mathbf{P}', \mathbf{P}) = \lim_{\lambda \to 0} \int d\mathbf{x}\, e^{-i\tau(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}-\lambda|\mathbf{x}|} \psi_{\mathbf{k}'}^{\mathrm{C}*}(\mathbf{x}) \psi_{\mathbf{k}}^{\mathrm{C}}(\mathbf{x}). \tag{38}$$

We start by considering the matrix element of the $\mathscr{C}\mathscr{F}$-transformed short-range potential $\mathscr{V}_\alpha = \mathscr{F}_{\mathrm{C}}^\dagger V_\alpha \mathscr{F}_{\mathrm{C}}$, Eq. (9), between trial functions $G_i \in C_0^\infty(\mathbf{R}^6),\ i = 1, 2$.

$$
\begin{aligned}
I_0 \;:=\; & \langle G_2 | \mathscr{F}_{\mathrm{C}}^{\dagger} V_\alpha \mathscr{F}_{\mathrm{C}} | G_1 \rangle \\
= \; & \int d\mathbf{k}_1' \, d\mathbf{p}_1' \, G_2^*(\mathbf{k}_1', \mathbf{p}_1') \frac{1}{(2\pi)^{3/2}} \int d\mathbf{x}_1 \, d\mathbf{y}_1 \, e^{-i\mathbf{p}_1' \cdot \mathbf{y}_1} \psi_{\mathbf{k}_1'}^{\mathrm{C}*}(\mathbf{x}_1) \\
& \times V_\alpha(\mathbf{x}_\alpha) \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k}_1 \, d\mathbf{p}_1 \, \psi_{\mathbf{k}_1}^{\mathrm{C}}(\mathbf{x}_1) e^{i\mathbf{p}_1 \cdot \mathbf{y}_1} G_1(\mathbf{k}_1, \mathbf{p}_1).
\end{aligned} \tag{39}
$$

It is obvious that the singularities of (38) are due to the divergence of the integral for large $|\mathbf{x}|$ in the limit $\lambda \to 0$. Thus, its main singular part can be isolated by replacing the Coulomb wave functions $\psi_{\mathbf{k}}^{\mathrm{C}}(\mathbf{x})$ and $\psi_{\mathbf{k}'}^{\mathrm{C}*}(\mathbf{x})$ by their weak asymptotics; that is, instead of (38) it suffices to investigate

$$
J_\lambda^{(\mathrm{as})} := \int d\mathbf{x} \, e^{-i\tau(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x} - \lambda|\mathbf{x}|} \psi_{\mathbf{k}'}^{(\mathrm{as})*}(\mathbf{x}) \, \psi_{\mathbf{k}}^{(\mathrm{as})}(\mathbf{x}), \tag{40}
$$

where

$$\psi_{\mathbf{k}}^{(\mathrm{as})}(\mathbf{x}) \;=\; \frac{i}{(2\pi)^{1/2}k}\left\{\delta(\hat{\mathbf{x}}+\hat{\mathbf{k}})\frac{e^{-ikx-iw_0(x)}}{x}\right.$$
$$\left. -s_{\mathrm{c}}(k,\hat{\mathbf{x}},\hat{\mathbf{k}})\frac{e^{ikx+iw_0(x)}}{x}\right\}. \tag{41}$$

The phase $w_0(x)$ is given as $w_0(x) \equiv w_0(k,x) = -\gamma \log 2kx$, and the Coulomb S-matrix $s_{\mathrm{c}}(k,\hat{\mathbf{x}},\hat{\mathbf{k}})$ as

$$s_{\mathrm{c}}(k,\hat{\mathbf{x}},\hat{\mathbf{k}}) = \frac{2^{1+2i\gamma}\,\gamma}{2i\pi}\frac{e^{2i\sigma_0}}{|\hat{\mathbf{x}}-\hat{\mathbf{k}}|^{2+i2\gamma}}, \tag{42}$$

with $\sigma_0 = \arg\Gamma(1+i\gamma)$.

We point out that Eq. (41) is the distribution version of the asymptotics as $|\mathbf{x}| \to \infty$, of the integral

$$
\begin{aligned}
\int d\Omega_{\mathbf{k}}\, \psi^{\mathrm{C}}_{\mathbf{k}}(\mathbf{x}) g(\hat{\mathbf{k}}) \overset{|\mathbf{x}|\to\infty}{\sim} & \int d\Omega_{\mathbf{k}}\, \psi^{(\mathrm{as})}_{\mathbf{k}}(\mathbf{x}) g(\hat{\mathbf{k}}) \\
= & \frac{i}{(2\pi)^{1/2} k} \left\{ g(-\hat{\mathbf{x}}) \frac{e^{-ikx - iw_0(x)}}{x} \right. \\
& \left. - \langle s_{\mathrm{c}}(k, \hat{\mathbf{x}}, \cdot), g \rangle \frac{e^{ikx + iw_0(x)}}{x} \right\}
\end{aligned} \tag{43}
$$

(for a description of the general procedure of how to evaluate the asymptotic behavior of integrals of such type see

L. D. Faddeev and S. P. Merkuriev, Quantum Scattering Theory for Several Particle Systems, (Kluwer, Dordrecht, 1993).

Here, $g(\hat{\mathbf{k}})$ is a smooth function on the unit sphere, and $\langle s_{\mathrm{c}}(k,\hat{\mathbf{x}},\cdot),g\rangle$ denotes the action of Coulomb S-matrix considered as a distribution, i. e.,

$$\langle s_{\mathrm{c}}(k,\hat{\mathbf{x}},\cdot),g\rangle := e^{2i\sigma_0}g(\hat{\mathbf{x}}) + \frac{2^{i\gamma}\,\gamma e^{2i\sigma_0}}{2i\pi}\int_0^2 dt\,\frac{G(t)-G(0)}{t^{1+i\gamma}}, \tag{44}$$

with

$$G(0) = 2\pi g(\hat{\mathbf{x}}),\; G(t) = \int_0^{2\pi} d\phi\, g(t,\phi). \tag{45}$$

Similarly, the formula

$$s_{\mathrm{c}}(k,\hat{\mathbf{x}},\hat{\mathbf{k}}) = e^{2i\sigma_0}\delta(\hat{\mathbf{x}}-\hat{\mathbf{k}}) + s_{\mathrm{c}}^{\mathrm{p}}(k,\hat{\mathbf{x}},\hat{\mathbf{k}}) \tag{46}$$

will be used as symbolic representation of (44).

It is apparent that in the quantities $J^{(\mathrm{as})}_{\lambda,k}$ (40), for $k = 1,2,3$, the integrals over x are of similar type. They can be evaluated by means of

$$\int_0^\infty dx\, x^{ia} e^{\pm itx - \lambda x} = e^{\pm i\pi/2 \mp \pi a/2} \frac{\Gamma(1+ia)}{(t \pm i\lambda)^{1+ia}}. \tag{47}$$

For the investigation of the limit $\lambda \to 0$ we have to introduce a new distribution which corresponds to the limit $\lambda \to 0$ of the denominator $(t \pm i\lambda)^{1+ia}$ in (47). This distribution is defined by its action on a smooth function $g(t)$ as

$$\begin{aligned}
\langle (t \pm i0)^{-1-ia}, g \rangle \; &:= \; \lim_{\lambda \to 0} \int_{-\infty}^{\infty} \frac{dt\, g(t)}{(t \pm i\lambda)^{1+ia}} \\
&= \; \frac{g(0)}{-ia}[1 - e^{\pm \pi a}] \\
&\quad + \int_{-1}^{1} \frac{dt\,[g(t) - g(0)]}{(t \pm i0)^{1+ia}} \\
&\quad + \int_{-\infty}^{-1} \frac{dt\, g(t)}{(t \pm i0)^{1+ia}} \\
&\quad + \int_{1}^{\infty} \frac{dt\, g(t)}{(t \pm i0)^{1+ia}}.
\end{aligned} \tag{48}$$

Symbolically it is written as

$$(t \pm i0)^{-1-ia} = \frac{i}{a}\left(1 - e^{\pm \pi a}\right) \delta(t) + \mathscr{P}(t \pm i0)^{-1-ia} \tag{49}$$

where $\mathscr{P} \cdots$ denotes the integral expressions on right hand side of (48).

It is shown that the leading singular term of the function $\mathscr{L}_\alpha^{\mathrm{C}}(\mathbf{P}', \mathbf{P})$ can be separated off as

$$\begin{aligned}\mathscr{L}_\alpha^{\mathrm{C}}(\mathbf{P}', \mathbf{P}) \;=\; & \boldsymbol{\Omega}_{\mathrm{s}}(\mathbf{k}_1', \mathbf{k}_1)\, \delta((\mathbf{p}_\alpha - \mathbf{p}_\alpha') \cdot \hat{\mathbf{k}}_1) \\ & + \boldsymbol{\Omega}_{\mathrm{r}}(\mathbf{P}', \mathbf{P}, \mathbf{p}_\alpha - \mathbf{p}_\alpha'). \end{aligned} \tag{50}$$

The Coulomb distorting factor $\boldsymbol{\Omega}_{\mathrm{s}}$ which multiplies the leading singular part has the form

$$\begin{aligned}\boldsymbol{\Omega}_{\mathrm{s}}(\mathbf{k}_1', \mathbf{k}_1) \;=\; & |s_{\alpha 1}| \frac{e^{i(\sigma_0 - \sigma_0')}}{k_1 k_1'} \Big\{ \boldsymbol{\Omega}_{\mathrm{s}}^{(-)}(\mathbf{k}_1', \mathbf{k}_1) \delta(\hat{\mathbf{k}}_1 - \hat{\mathbf{k}}_1') \\ & - \boldsymbol{\Omega}_{\mathrm{s}}^{(+)}(\mathbf{k}_1', \mathbf{k}_1) \delta(\hat{\mathbf{k}}_1 + \hat{\mathbf{k}}_1') \Big\} \end{aligned} \tag{51}$$

where

$$\Omega_{\mathrm{s}}^{(-)}(\mathbf{k}'_1,\mathbf{k}_1) := \frac{\sinh[\pi(\gamma_1-\gamma'_1)/2]}{\pi(\gamma_1-\gamma'_1)/2}\Re\Big\{(2k_1)^{i\gamma_1}(2k'_1)^{-i\gamma'_1} \times\Gamma(1+i(\gamma_1-\gamma'_1))e^{-i(\sigma_0-\sigma'_0)}\Big\} \tag{52a}$$

$$\Omega_{\mathrm{s}}^{(+)}(\mathbf{k}_1,\mathbf{k}'_1) := \frac{\sinh[\pi(\gamma_1+\gamma'_1)/2]}{\pi(\gamma_1+\gamma'_1)/2}\Re\Big\{(2k_1)^{i\gamma_1}(2k'_1)^{i\gamma'_1} \times\Gamma(1+i(\gamma_1+\gamma'_1))e^{-i(\sigma_0+\sigma'_0)}\Big\}. \tag{52b}$$

Here, $\Re$ means real part, $\sigma_0 \equiv \sigma_0(k_1) = \arg\Gamma(1+i\gamma_1)$ and similarly for $\sigma'_0 \equiv \sigma_0(k'_1)$, with $\gamma'_1 = n_1/2k'_1$, and $\delta(\hat{\mathbf{k}}'\pm\hat{\mathbf{k}})$ is the δ-function on the unit sphere. The second term $\Omega_{\mathrm{r}}(\mathbf{P},\mathbf{P}',\mathbf{p}_\alpha-\mathbf{p}'_\alpha)$ contains distributions which are less singular than those of the first term.

Derivation of the most singular part of $\mathscr{L}_c(\mathbf{P}', \mathbf{P})$.
2. Partial analysis method

To study the most singular term of the distribution

$$\mathscr{L}(\mathbf{k}, \mathbf{k}'; \mathbf{q}) = \int d\mathbf{r} \psi^c(\mathbf{k}, \mathbf{r}) e^{-i<\mathbf{q},\mathbf{r}>} \psi^{c*}(\mathbf{k}', \mathbf{r}), \tag{53}$$

we are using the two-body Coulomb wave function $\psi^c(\mathbf{k}, \mathbf{r})$ definition

$$\psi^c(\mathbf{k}, \mathbf{r}) = (2\pi)^{-3/2} e^{-\pi\gamma/2} \Gamma(1 + i\gamma) e^{i<\mathbf{k},\mathbf{r}>} \Phi(-i\gamma, 1, i\zeta),$$

where $\gamma = \frac{n_c}{2k}$, $\zeta = kr - <\mathbf{k}, \mathbf{r}>$.
We collect all exponential terms together and rewrite the rest of initial expression (53) as

$$\mathscr{L}(\mathbf{k}, \mathbf{k}'; \mathbf{q}) = (2\pi)^{-3} e^{-\frac{\pi(\gamma+\gamma')}{2}} \Gamma(1 + i\gamma) \Gamma(1 - i\gamma') \times \tag{54}$$

$$\times \int d\mathbf{r} e^{i<\boldsymbol{\omega},\mathbf{r}>} \Phi(-i\gamma, 1, i\zeta) \Phi(i\gamma', 1, -i\zeta'),$$

where as above $\gamma' = \frac{n_c}{2k'}$, $\zeta' = k'r - <\mathbf{k}', \mathbf{r}>$ and $\boldsymbol{\omega} = \mathbf{k} - \mathbf{k}' - \mathbf{q}$.

Next step is to make a partial decomposition for all three functions under the integral. Let us start with confluent hypergeometric function $\Phi(-i\gamma, 1, i(kr - <\mathbf{k}, \mathbf{r}>))$.

First we derive the coefficients $\Phi_l(k,r)$ of the set

$$\Phi(-i\gamma, 1, i(kr - <\mathbf{k}, \mathbf{r}>)) = \sum_{l=0}^{\infty}(2l+1)\Phi_l(k,r)P_l(\cos\theta),$$

$$\cos\theta = <\hat{\mathbf{k}}, \hat{\mathbf{r}}>,$$

which is

$$\Phi_l(k,r) = \frac{\Gamma(l-i\gamma)}{\Gamma(-i\gamma)\Gamma(2l+2)}(-2ikr)^l\Phi(l-i\gamma, 2l+2, 2ikr). \tag{55}$$

Now we can write the parcial decompositions for all functions under the integral (54).

For the exponent the following expansion is valid

$$e^{i<\omega,\mathbf{r}>} = \frac{4\pi}{\omega r}\sum_{l,m} i^l u_l(\omega r) Y_l^m(\hat{\mathbf{r}}) Y_l^{m*}(\hat{\boldsymbol{\omega}}), \tag{56}$$

where

$$u_l(z) = \sqrt{\frac{\pi z}{2}} J_{l+\frac{1}{2}}(z)$$

and J_p are the Bessel functions. Note, we should keep the full decomposition in (56) even at arbitrary large parameter r, because we are studing the most singular part of the integral (54) in vicinity of the point $\omega = 0$. It means that the multiplication ωr is an arbitrary value.

Since the integrand in (54) is regular at arbitrary $\mathbf{r} \in \mathbf{R}^3$, the unlimited integration domain at $r \to \infty$ will only contribute to the most singular term of that distribution. Therefore on the physical sheet of energy we use the asymptotic form

for the functions $\Phi(l - i\gamma, 2l+1, 2ikr)$ in (55)

$$\Phi(a,b,z) = \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}[1 + O(|z|^{-1})], \quad \Re(z) < 0,$$

which produces

$$\Phi(l-i\gamma,2l+2,2ikr)=\frac{\Gamma(2l+2)}{\Gamma(l+2+i\gamma)}(-2ikr)^{i\gamma-l}\left[1+O\left(\frac{1}{kr}\right)\right] \quad (57)$$

at arbitrary k, limited from below.

Conclusion

The most singular part of the integral (58) should be written as

$$\mathscr{L}(\mathbf{k},\mathbf{k}';\mathbf{q}) = \int d\mathbf{r}\,\psi^{c}(\mathbf{k},\mathbf{r})e^{-i<\mathbf{q},\mathbf{r}>}\psi^{c*}(\mathbf{k}',\mathbf{r}) \simeq \tag{58}$$

$$\simeq \mathscr{A}(\mathbf{k},\mathbf{k}')\delta(\mathbf{k}-\mathbf{k}'-\mathbf{q}),$$

where

$$\mathscr{A}(\mathbf{k},\mathbf{k}') = (2k)^{i\gamma}(2k')^{-i\gamma'}\Gamma(1+i\gamma)\Gamma(1-i\gamma')\times \tag{59}$$

$$\times\frac{\sinh\left\{\frac{\pi}{2}(\gamma-\gamma')\right\}}{\frac{\pi}{2}(\gamma-\gamma')}\,{}_2F_1\left(-i\gamma, i\gamma'; 1; \frac{1+<\hat{\mathbf{k}},\hat{\mathbf{k}}'>}{2}\right).$$

For the special case $\hat{\mathbf{k}}_1 = \hat{\mathbf{k}}_1'$

$${}_2F_1\left(-i\gamma, i\gamma'; 1; \frac{1+<\hat{\mathbf{k}},\hat{\mathbf{k}}'>}{2}\right) \underset{\hat{\mathbf{k}}\to\hat{\mathbf{k}}'}{\longrightarrow} \frac{\Gamma(1+i(\gamma-\gamma'))}{\Gamma(1+i\gamma)\Gamma(1-i\gamma')},$$

so the derived result almost coincide with the first term of expression, obtained by the weak asymptotic method.

One can easily check the properties

$$\mathscr{A}(\mathbf{k},\mathbf{k}) = 1 \tag{60}$$

$$\mathscr{A}(\mathbf{k},\mathbf{k}') \underset{k,k'\to\infty}{\longrightarrow} 1, \qquad \mathscr{A}(\mathbf{k},\mathbf{k}') \underset{\gamma,\gamma'\to 0}{\longrightarrow} 1. \tag{61}$$

Fynally we calculated the most singular part of the Nordsieck type integral (58) which playing also an important role in a theory of representations in atomic physics.

Both methods predict the same asymptotic behavior of the potential V_α in configuration space.