The space-time symmetry group of a spin $1/2$ elementary particle

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Atomistic hypothesis
Matter cannot be divided indefinitely. After a finite number of steps we reach a final and indivisible object. We call it an **elementary particle**.

Definition: An **elementary particle** is a mechanical system without excited states. We can destroy it but we can never modify its structure. All its possible states are only kinematical modifications of any one of them.

If the state of an elementary particle changes, it is always possible to find another inertial observer who describes the particle in the same state as in the previous instant.

Corollary

The kinematical space of an elementary particle is necessarily a homogeneous space of the kinematical group associated to the restricted Relativity Principle.

The kinematical variables are $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha})$, which are interpreted as the time, position of the charge, velocity of the charge and orientation of the system. The system which satisfies Dirac equation is such that $u = c$.

These 9 variables are the non compact variables $t, \boldsymbol{r}$ and the dimensionless compact variables $\widetilde{\theta}, \widetilde{\phi}$ which represent the direction of the velocity $\boldsymbol{u}$ and the α, θ, ϕ which represent a normal parameterization of the orientation of the body frame $\boldsymbol{e}_i$.

Dirac equation corresponds to the classical relationship

$$H - \boldsymbol{u} \cdot \boldsymbol{P} - \boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u} \right) = 0.$$

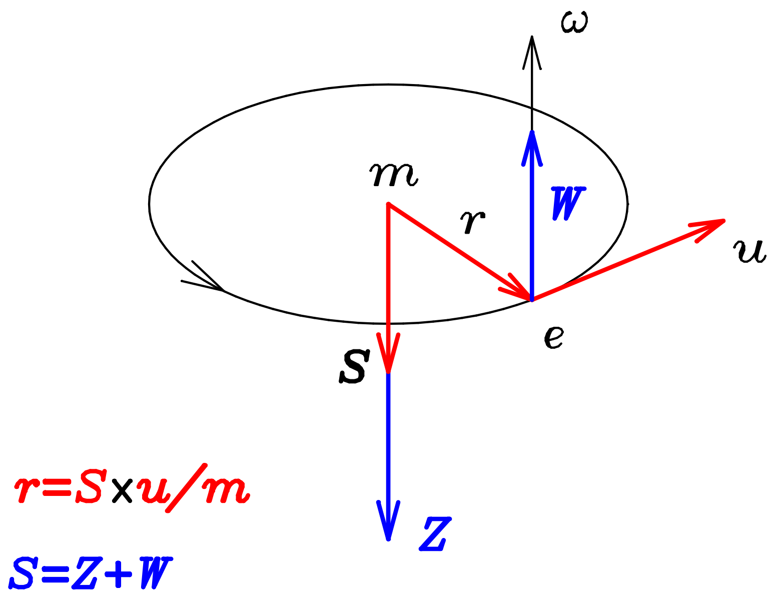
where the spin has a twofold structure

$$\boldsymbol{S} = \boldsymbol{u} \times \frac{\partial L}{\partial \dot{\boldsymbol{u}}} + \frac{\partial L}{\partial \boldsymbol{\omega}} = \boldsymbol{Z} + \boldsymbol{W}.$$

- $\boldsymbol{Z}$ is the Zitterbewegung part of the spin

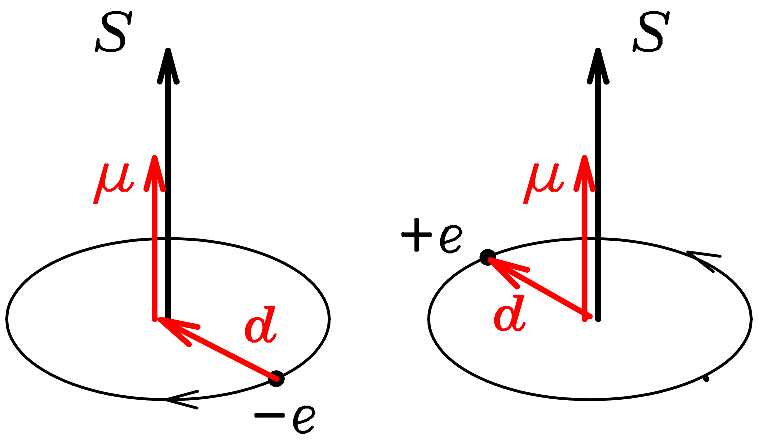
- $\boldsymbol{W}$ is the Rotational part of the spin

A Dirac particle



Total spin $\boldsymbol{S} = \boldsymbol{Z}+\boldsymbol{W}$, is $1/2$ and is pointing in the same direction as the zitterbewegung part $\boldsymbol{Z}$ which quantizes with $z = 0, 1$.

Chirality



$a)$ $b)$

Particle and antiparticle have the same mass and spin and also the same electric and magnetic dipole with the same relative orientation with respect to the spin.

Matter is lefthanded and antimatter is righthanded.

When quantizing the system the wave fuction is a function of the above variables

$$\Phi(t,\boldsymbol{r};\widetilde{\theta},\widetilde{\phi},\alpha,\theta,\phi)=\sum_{i=1}^{i=4}\psi_i(t,\boldsymbol{r})\Phi_i(\widetilde{\theta},\widetilde{\phi},\alpha,\theta,\phi)$$

If the system has spin S and mass m we can define a length scale $R=S/mc$ and a time scale $T=S/mc^2$, so that the 9 kinematical variables of a Dirac particle can be taken dimensionless from the classical point of view.

This means that this system, in addition to the Poincaré group, it also has as a symmetry group the space-time dilations which do not change the speed of light

$$t'=e^{\lambda}t,\quad \boldsymbol{r}'=e^{\lambda}\boldsymbol{r},\quad \boldsymbol{u}'=\boldsymbol{u},\quad \boldsymbol{\alpha}'=\boldsymbol{\alpha}.$$

the generator of the unitary representation of this $U(1)$ group is

$$D = it\frac{\partial}{\partial t} + i\boldsymbol{r}\cdot\nabla = tH - \boldsymbol{r}\cdot\boldsymbol{P}.$$

The enlarged group is sometimes called the Weyl group. We shall denote it by $\mathcal{P}_D$.

The Poincaré group $\mathcal{P}$ has two Casimir operators

$$C_1 = P^\mu P_\mu = H^2 - \boldsymbol{P}^2 = m^2, \quad C_2 = -W^\mu W_\mu = m^2 s(s+1)\hbar^2.$$

The enlarged Weyl group $\mathcal{P}_D$ has only one Casimir operator which for massive systems is reduced to

$$C = \frac{C_2}{C_1} = s(s+1)\hbar^2.$$

SPIN is the only intrinsic property of this elementary particle.

The rotation group acts on the kinematical variables in the way:

$$t' = t, \quad \boldsymbol{r}' = R(\boldsymbol{\mu})\boldsymbol{r}, \quad \boldsymbol{u}' = R(\boldsymbol{\mu})\boldsymbol{u}, \quad R(\boldsymbol{\alpha}') = R(\boldsymbol{\mu})R(\boldsymbol{\alpha}).$$

But the orientation variables $\boldsymbol{\alpha}$ are arbitrary, so that we can also have another local rotation body frame transformations

$$t' = t, \quad \boldsymbol{r}' = \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad R(\boldsymbol{\alpha}') = R(\beta)R(\boldsymbol{\alpha}).$$

This corresponds to the active rotation of the body frame $\boldsymbol{e}_i$, $i = 1, 2, 3$. The generators of these rotations are the spin projections on the body frame, *i.e.* the $T_i = \boldsymbol{e}_i \cdot \boldsymbol{W}$ operators. These operators commute with the whole $\mathcal{P}_D$ group. So that the new space-time group is just

$$\mathcal{P}_D \otimes SU(2).$$

It has two Casimir operators $\boldsymbol{S}^2$ and $\boldsymbol{T}^2$. But because $\boldsymbol{T}^2 = \boldsymbol{W}^2$ the eigenvalues of $\boldsymbol{T}^2$ are only $1/2$.

The T_i operators satisfy the commutation relations

$$[T_i, T_j] = -i\epsilon_{ijk}T_k$$

wich corresponds to an active rotation.

Because the spin has the form $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$ and quantizes with $s = 1/2$ while $\boldsymbol{W}$ quantizes with $w = 1/2$, the zitterbewegung part $\boldsymbol{Z}$ quantizes with $z = 0$ or $z = 1$.

There are thus two kinds of Dirac's particles according to the $\boldsymbol{Z}$ eigenvalue.

Discussion

If we interpret the new $SU(2)$ local rotation group as representing ISOSPIN and the zitterbewegung angular momentum $\boldsymbol{Z}$ as representing COLOUR, the above Dirac particle with the $\boldsymbol{P}_D \otimes SU(2)$ as its space-time symmetry group is:

- A particle of spin 1/2 and of isospin 1/2 of arbitrary nonvanishing mass and arbitrary charge.

- which can be in a colourless state $z = 0$ (lepton?), or in a coloured state $z = 1$ (quark?). It can also be in one of the three Z_3 states $1, 0, -1$ but the Z_3 is unobservable because the four basic $\Phi_i(\widetilde{\theta}, \widetilde{\phi}, \alpha, \theta, \phi)$ spinors are eigenvectors of $\boldsymbol{Z}^2$, $\boldsymbol{S}^2$ and $\boldsymbol{T}^2$ but not of Z_3 for $z = 1$ case.

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