DESCRIPTION
OF PARTICLE SPIN MOTION
IN STORAGE RINGS
AND QUANTUM EFFECTS

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1 INTRODUCTION

It is difficult to separate semiclassical and quantum methods of description of spin dynamics because any spin effect is quantum. The semiclassical method consists in averaging spin operators and using an average spin. Such an approach is presented, for example, by the well-known Thomas-Bargmann-Michel-Telegdi (T-BMT) equation. We also use one of purely quantum methods and calculate an evolution of a two-component spin wave function. Of course, every method should give the same result. As distinct from other works, we use the cylindrical coordinate system. This system can be helpful if the ring is circular and we need to obtain the analytical solution of problem.

2 GENERAL EQUATIONS OF PARTICLE AND SPIN MOTION

The Thomas-Bargmann-Michel-Telegdi (T-BMT) equation uses the Cartesian coordinates. The transformation of the T-BMT equation to the cylindrical coordinates should be performed with an allowance for oscillatory terms in the particle motion equation.

The particle motion is described by the Lorentz equation

$$\frac{d\boldsymbol{p}}{dt} = e\left(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}\right), \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}.$$

It is convenient to use the unit vector of momentum direction, $\boldsymbol{N} = \boldsymbol{p}/p$, which defines the direction of particle motion. The equation of particle motion takes the form

$$\frac{d\boldsymbol{N}}{dt} = \boldsymbol{\omega} \times \boldsymbol{N},$$
$$\boldsymbol{\omega} = -\frac{e}{\gamma m}\left(\boldsymbol{B} - \frac{\boldsymbol{N} \times \boldsymbol{E}}{\beta}\right),$$

where $\boldsymbol{\omega}$ is the angular velocity of the particle rotation.

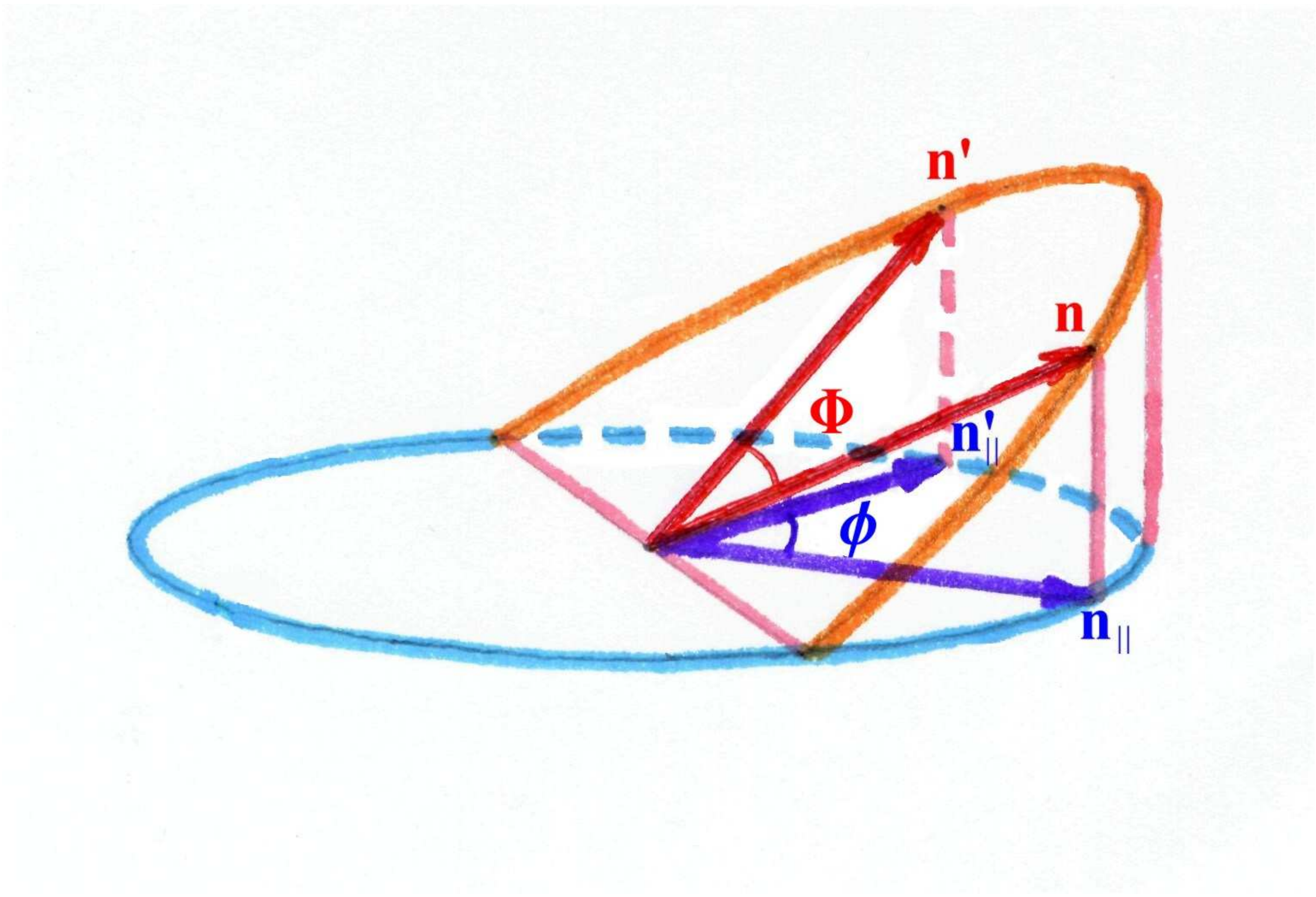
The spin motion is determined by the T-BMT equation

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega}_{T-BMT} \times \boldsymbol{s},$$
$$\boldsymbol{\Omega}_{T-BMT} = -\frac{e}{2m}\{\left(g - 2 + \frac{2}{\gamma}\right)\boldsymbol{B} \tag{1}$$
$$-\frac{(g-2)\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B}) - \left(g - 2 + \frac{2}{\gamma+1}\right)(\boldsymbol{\beta}\times\boldsymbol{E})\},$$

where $\boldsymbol{s}$ is the spin vector, that is the expectation value of the quantum mechanical spin operator.

3 CORRECTIONS TO THE PARTICLE MOTION IN THE HORIZONTAL PLANE

Vertical betatron oscillations and orbit distortions change the plane of the particle motion. The (pseudo)vector of angular velocity becomes tilted.



The angle Φ between two positions of rotating vector $\boldsymbol{N}$ in the tilted plane is not equal to the angle ϕ between two corresponding horizontal projections. Therefore, the vertical betatron oscillations and orbit distortions change the instantaneous angular velocity of particle motion. This effect can be calculated.

The instantaneous angular velocity of particle rotation in the horizontal plane is given by

$$\dot{\phi} \equiv \frac{d\phi}{dt} = \frac{(\boldsymbol{N}_{\|} \times \dot{\boldsymbol{N}}_{\|}) \cdot \boldsymbol{e}_z}{|\boldsymbol{N}_{\|}|^2} = \omega_z - o, \tag{2}$$

$$o = \frac{(\omega_x N_x + \omega_y N_y) N_z}{1 - N_z^2} = \frac{(\omega_\rho N_\rho + \omega_\phi N_\phi) N_z}{1 - N_z^2}. \tag{3}$$

Eqs. (2),(3) are exact. The validity of these equations can be confirmed.

The correction o is usually small and even negligible. The particle momentum deflection is given by

$$N_\rho = \frac{p_\rho}{p} = \rho_0 \sin(\omega_r t + \alpha),$$
$$N_z = \frac{p_z}{p} = \psi_0 \sin(\omega_v t + \beta),$$

where ρ_0 and ψ_0 are the angular amplitudes of radial and vertical CBOs.

With an allowance for the orders of quantities

$$\omega_\rho \sim \psi_0, \quad \omega_\phi \sim \left\{ \begin{array}{l} \rho_0^2 \\ \psi_0^2 \end{array} \right. ,$$
$$N_\rho \sim \rho_0, \quad N_\phi \approx \pm 1, \quad N_z \sim \psi_0,$$

we obtain that the quantity o is of the third order in the angular amplitudes ρ_0 and ψ_0. Moreover, it oscillates and therefore it averages to zero. If we take into account only second-order terms in the angular amplitudes and the average particle orbit is not tilted, the quantity o is negligible. Approximately,

$$\dot{\phi} = \omega_z = -\frac{e}{\gamma m}\left(B_z - \frac{(\boldsymbol{N} \times \boldsymbol{E})_z}{\beta}\right). \tag{4}$$

4 EQUATION OF SPIN MOTION IN STORAGE RINGS

The transformation of the T-BMT equation to the cylindrical coordinate system leads to the equation

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\omega}_a \times \boldsymbol{s}, \quad \boldsymbol{\omega}_a = \boldsymbol{\Omega} - \dot{\phi}\boldsymbol{e}_z. \tag{5}$$

In this equation, $\boldsymbol{\omega}_a$ is the the angular velocity of spin rotation in the cylindrical coordinates. The corresponding angular velocity in the Cartesian coordinates equals $\boldsymbol{\Omega}$. The difference between these quantities is caused by the rotation of the axes $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\phi$.

Formulae for the electric dipole moment (EDM) can be obtained from the corresponding formulae for the anomalous magnetic moment with the substitution $\boldsymbol{B} \rightarrow \boldsymbol{E}$, $\boldsymbol{E} \rightarrow -\boldsymbol{B}$, $g-2 \rightarrow \eta$.

The allowance for the particle EDM leads to the modification of the T-BMT equation that takes the form

$$\begin{aligned}
\frac{d\boldsymbol{s}}{dt} &= \boldsymbol{\Omega}\times\boldsymbol{s}, \quad \boldsymbol{\Omega}=\boldsymbol{\Omega}_{T-BMT}+\boldsymbol{\Omega}_{EDM},\\
\boldsymbol{\Omega}_{T-BMT} &= -\frac{e}{m}\left[\left(a+\frac{1}{\gamma}\right)\boldsymbol{B}-\frac{a\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\right.\\
&\quad\left.-\left(a+\frac{1}{\gamma+1}\right)(\boldsymbol{\beta}\times\boldsymbol{E})\right],\\
\boldsymbol{\Omega}_{EDM} &= -\frac{e\eta}{2m}\left(\boldsymbol{E}-\frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E})+\boldsymbol{\beta}\times\boldsymbol{B}\right), \qquad (6)
\end{aligned}$$

where $\boldsymbol{\Omega}_{T-BMT}$ is defined by Eq. (1), $a=(g-2)/2$, and $\eta = 4dm/e$.

As a rule, we can neglect the term $\frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E})$.

This formula for the angular velocity of spin rotation in the cylindrical coordinates is exact:

$$\boldsymbol{\omega}_a = -\frac{e}{m}\Biggl\{a\boldsymbol{B} - \frac{a\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})$$
$$+\left(\frac{1}{\gamma^2-1}-a\right)(\boldsymbol{\beta}\times\boldsymbol{E}) + \frac{1}{\gamma}\left[\boldsymbol{B}_{\|} - \frac{1}{\beta^2}(\boldsymbol{\beta}\times\boldsymbol{E})_{\|}\right]$$
$$+\frac{\eta}{2}\left(\boldsymbol{E} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E}) + \boldsymbol{\beta}\times\boldsymbol{B}\right)\Biggr\} + o\boldsymbol{e}_z. \qquad (7)$$

After neglecting small terms, this equation takes the form

$$\boldsymbol{\omega}_a = -\frac{e}{m}\Biggl\{a\boldsymbol{B} - \frac{a\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B}) + \left(\frac{1}{\gamma^2-1}-a\right)(\boldsymbol{\beta}\times\boldsymbol{E})$$
$$+\frac{1}{\gamma}\left[\boldsymbol{B}_{\|} - \frac{1}{\beta^2}(\boldsymbol{\beta}\times\boldsymbol{E})_{\|}\right] + \frac{\eta}{2}(\boldsymbol{E}+\boldsymbol{\beta}\times\boldsymbol{B})\Biggr\}. \qquad (8)$$

5 EFFECT OF VERTICAL BETATRON OSCILLATIONS ON SPIN DYNAMICS

In the g−2 experiment, the correction for the vertical BO (pitch) is about 0.2 ppm. This correction is very important. The effect has been described by F. Farley
[F.J.M. Farley, Phys. Lett. B **42**, 66 (1972).]

The result has been confirmed by J. Field and G. Fiorentini
[J.H. Field and G. Fiorentini, Nuovo Cim. A**21**, 297 (1974)]
and by computer simulations.

The horizontal BO (yaw) does not give any significant corrections.

The theory of spin oscillations in the g−2 experiment can be developed in the very general form. The spin motion perturbed by the vertical BO is described by the equation

$$\frac{d\boldsymbol{s}}{dt} = \left\{a_0 + a_3 \cos\left[2(\omega_p t + \phi_p)\right]\right\}(\boldsymbol{e}_3 \times \boldsymbol{s})$$
$$+a_2 \cos(\omega_p t + \phi_p)(\boldsymbol{e}_2 \times \boldsymbol{s}) + a_1 \sin(\omega_p t + \phi_p)(\boldsymbol{e}_1 \times \boldsymbol{s}),$$

where a_1 and a_2 are first-order quantities and a_3 is a second-order quantity in the angular amplitude of pitch, ψ_0. The quantity ω_p is the angular frequency of pitch.

If the pitch correction is small, it can be determined exactly:

$$\omega_a =< \dot{\phi} >= a_0 + \frac{a_0(a_1^2 + a_2^2) - 2a_1a_2\omega_p}{4(a_0^2 - \omega_p^2)}$$
$$+\frac{a_0(a_1^2 - a_2^2)}{4(a_0^2 - \omega_p^2)}\langle\cos 2(a_0 t + \phi_0)\rangle \cdot \frac{1 + s_\perp^2}{1 - s_\perp^2}.$$

In preceding works, both electric and magnetic focusing have been considered. The coefficients a_i $(i = 0, 1, 2, 3)$ are equal to

$$a_0 = \lambda\omega_0\left(1 - \frac{\gamma - 1}{2\gamma}\psi_0^2\right), \quad a_1 = -\omega_0\frac{\gamma - 1}{\gamma}\psi_0,$$
$$a_2 = -\lambda f\omega_p\psi_0, \quad a_3 = \lambda\omega_0\frac{\gamma - 1}{2\gamma}\psi_0^2,$$

where λ equals 1 and -1 for negative and positive muons, respectively. The factor f is

$$f = 1 + a\gamma - \frac{1 + a}{\gamma} = 1 + a\beta^2\gamma - \frac{1}{\gamma}$$

and

$$f = 1 + a\gamma$$

for electric and magnetic focusing, respectively.

The g−2 frequency equals

$$
\begin{aligned}
\omega_a &= \omega_0(1-C), \\
C &= \frac{1}{4}\psi_0^2\left[1 - \frac{\omega_0^2}{\gamma^2(\omega_0^2-\omega_p^2)} - \frac{\omega_p^2(f-1)(f-1+2/\gamma)}{\omega_0^2-\omega_p^2}\right. \\
&\left. - \frac{(\gamma-1)^2\omega_0^2 - f^2\gamma^2\omega_p^2}{\gamma^2(\omega_0^2-\omega_p^2)} < \cos\left[2(\omega_0 t+\phi_0)\right] > \cdot \frac{1+s_\perp^2}{1-s_\perp^2}\right].
\end{aligned} \tag{9}
$$

Formula (9) is in the best agreement with the result found by F. Farley
[F.J.M. Farley, Phys. Lett. B **42**, 66 (1972).]

However, it contains the additional oscillatory term which is zero on the average. It is possible to include the oscillatory term in a fitting process instead of its elimination. In the real g−2 experiment, $f=1,\ s_3^{(0)}=0,\ \gamma\gg 1,\ \phi_0$ equals 0 or π, and formula (9) takes the form

$$\omega_a = \omega_0(1 - C), \quad C = \frac{1}{4}\psi_0^2 \left[1 - < \cos\left(2\omega_0 t\right) >\right].$$

This formula shows the inclusion of the oscillatory term in a fitting process is not difficult. The vertical BOs violate the sinusoidality of the spin motion.

6 QUANTUM DESCRIPTION OF SPIN DYNAMICS NEAR RESONANCES

Perhaps, the best description of quantum spin dynamics was given in "The Feynman Lectures on Physics". We follow this approach.

The spin state of particle is defined by the two-component wave function

$$\Psi = \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix},$$

where $C_1(t)$ and $C_2(t)$ are time-dependent amplitudes. The wave functions $|u>=\begin{pmatrix}1\\0\end{pmatrix}$ and $|d>=\begin{pmatrix}0\\1\end{pmatrix}$ characterize the spin-up and spin-down states those energies are $E_1^{(0)}$ and $E_2^{(0)}$. We take into account a possible decay of the particle:

$$E_1^{(0)} = E_0 + \frac{\omega_0}{2} - i\frac{\Gamma}{2},$$
$$E_2^{(0)} = E_0 - \frac{\omega_0}{2} - i\frac{\Gamma}{2},$$

where $\Gamma = 1/\tau$ is a decay constant, τ is the lifetime, and ω_0 is the difference between the energies of the nonperturbed states. We use the system of units $\hbar = c = 1$.

The dynamics of the wave function Ψ is defined by

$$i\frac{d\Psi}{dt} = H\Psi, \quad H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \tag{10}$$
$$H_{ij} =< i|\mathcal{H}|j >, \quad i,j = 1,2.$$

Certainly, $H_{21} = H_{12}^*$.

Usual representation of the spin matrices leads to cumbersome calculations. We do an important change of definition of the Pauli matrices. The calculations are strongly simplified with the direct substitution of the Pauli matrices for σ_ρ and σ_ϕ :

$$\sigma_\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{11}$$

It can be grounded. Of course, such a substitution changes the form of the Hamiltonian, which should be consistent with the equation of spin motion in the cylindrical coordinates.

The Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}_a \qquad (12)$$

or

$$\mathcal{H} = -\frac{e}{2m}\left\{a\boldsymbol{\sigma}\cdot\boldsymbol{B} - \frac{a\gamma}{\gamma+1}(\boldsymbol{\sigma}\cdot\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\boldsymbol{B})\right.$$
$$+\left(\frac{1}{\gamma^2-1}-a\right)\boldsymbol{\sigma}\cdot(\boldsymbol{\beta}\times\boldsymbol{E}) + \frac{1}{\gamma}\left[\boldsymbol{\sigma}\cdot\boldsymbol{B}_\parallel - \frac{1}{\beta^2}\boldsymbol{\sigma}\cdot(\boldsymbol{\beta}\times\boldsymbol{E})_\parallel\right]$$
$$\left.+\frac{\eta}{2}\left[\boldsymbol{\sigma}\cdot\boldsymbol{E} - \frac{\gamma}{\gamma+1}(\boldsymbol{\sigma}\cdot\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\boldsymbol{E}) + \boldsymbol{\sigma}\cdot(\boldsymbol{\beta}\times\boldsymbol{B})\right]\right\} + \frac{1}{2}o\boldsymbol{\sigma}\cdot\boldsymbol{e}_z. \qquad (13)$$

The stated method can be verified by means of considering problems formerly solved by other methods. When the perturbation is caused by the longitudinal magnetic field B_ϕ, the calculated formula agree with the formerly obtained result
[A.J. Silenko, JETP, **87**, 629 (1998).]

7 SPIN DYNAMICS NEAR RESONANCES

We consider the spin dynamics near resonances when the beam energy is constant and $\omega \approx \omega_0$. As a result of averaging, terms oscillating with the angular frequency $\omega + \omega_0$ can be neglected.

The general solution of initial equation for the amplitudes of spin wave functions is given by

$$\mathcal{C}_1(t) = \left[\left(\cos\frac{\omega' t}{2} - i\frac{\omega_0 - \omega}{\omega'}\sin\frac{\omega' t}{2}\right)\mathcal{C}_1(0) - i\frac{2\mathcal{E}}{\omega'}\sin\frac{\omega' t}{2}\mathcal{C}_2(0)\right]e^{-\frac{\Gamma}{2}t},$$
$$\mathcal{C}_2(t) = \left[-i\frac{2\mathcal{E}^*}{\omega'}\sin\frac{\omega' t}{2}\mathcal{C}_1(0) + \left(\cos\frac{\omega' t}{2} + i\frac{\omega_0 - \omega}{\omega'}\sin\frac{\omega' t}{2}\right)\mathcal{C}_2(0)\right]e^{-\frac{\Gamma}{2}t}. \tag{14}$$

In this case
[J.H. Field and G. Fiorentini, Nuovo Cim. A**21**, 297 (1974)]

$$\omega' = \sqrt{(\omega_0 - \omega)^2 + 4\mathcal{E}_0^2}.$$

To calculate the angular frequency of spin rotation, ω_a, we have to take into account

i) $\omega_a \longrightarrow \omega_0$ when $\mathcal{E} \longrightarrow 0$;

ii) the spin state defined by Eq. (14) becomes equivalent to the initial spin state $\mathcal{C}_1(0), \mathcal{C}_2(0)$ when $t = 2\pi N/\omega'$ (N is integer).

The angular frequency of spin rotation is equal to

$$\omega_a = \omega + (\omega_0 - \omega)\sqrt{1 + \frac{4\mathcal{E}_0^2}{(\omega_0 - \omega)^2}}, \quad \omega_0 \neq \omega.$$

When $\omega_0 = \omega,\ \omega_a = \omega_0$. When the resonance is not perfect ($|\omega_0 - \omega| >> \mathcal{E}_0$),

$$\omega_a = \omega_0 + \frac{2\mathcal{E}_0^2}{\omega_0 - \omega}.$$

The periodical motion of spin near the resonance is very nonsinusoidal.

The problem was formerly solved by Field and Fiorentini

[J.H. Field and G. Fiorentini, Nuovo Cim. A**21**, 297 (1974)]

However, we cannot confirm the corresponding result obtained in this work. The discrepancy is caused by the disagreement of the approach used by Field and Fiorentini with condition "i)".

Let us consider two important cases.

INITIAL DIRECTION OF SPIN IS VERTICAL

$$C_1(0) = 1, \qquad C_2(0) = 0,$$
$$\mathcal{C}_1(t) = \left(\cos\frac{\omega' t}{2} - i\frac{\omega_0 - \omega}{\omega'}\sin\frac{\omega' t}{2}\right)e^{-\frac{\Gamma}{2}t},$$
$$\mathcal{C}_2(t) = -i\frac{2\mathcal{E}^*}{\omega'}\sin\frac{\omega' t}{2}e^{-\frac{\Gamma}{2}t}.$$

If the particle has not decayed, the probability to find it in the spin-flipped state is given by the known formula:

$$P(t) = \frac{|C_2(t)|^2}{|C_1(t)|^2 + |C_2(t)|^2} = \frac{4\mathcal{E}_0^2}{(\omega_0 - \omega)^2 + 4\mathcal{E}_0^2} \sin^2 \frac{\omega' t}{2}.$$

INITIAL DIRECTION OF SPIN IS HORIZONTAL

Let us consider the general case when the azimuth ψ defining an initial orientation of spin is arbitrary. In this case

$$
\begin{aligned}
C_1(0) &= \frac{1}{\sqrt{2}}e^{-i\psi/2}, \qquad C_2(0) = \frac{1}{\sqrt{2}}e^{i\psi/2}, \\
\mathcal{C}_1(t) &= \frac{1}{\sqrt{2}}e^{-i\psi/2}\left[\cos\frac{\omega' t}{2} + \frac{2\mathcal{E}_0\sin(\psi-\varphi)}{\omega'}\sin\frac{\omega' t}{2}\right. \\
&\left. -i\frac{\omega_0-\omega+2\mathcal{E}_0\cos(\psi-\varphi)}{\omega'}\sin\frac{\omega' t}{2}\right]e^{-\frac{\Gamma}{2}t}, \\
\mathcal{C}_2(t) &= \frac{1}{\sqrt{2}}e^{i\psi/2}\left[\cos\frac{\omega' t}{2} - \frac{2\mathcal{E}_0\sin(\psi-\varphi)}{\omega'}\sin\frac{\omega' t}{2}\right. \\
&\left. +i\frac{\omega_0-\omega-2\mathcal{E}_0\cos(\psi-\varphi)}{\omega'}\sin\frac{\omega' t}{2}\right]e^{-\frac{\Gamma}{2}t}
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
|C_1(t)|^2 &= \frac{1}{2}\Bigg[1 + \frac{2\mathcal{E}_0 \sin{(\psi-\varphi)}}{\omega'} \sin{(\omega' t)} \\
&+ \frac{4\mathcal{E}_0(\omega_0-\omega)\cos{(\psi-\varphi)}}{\omega'^2} \sin^2\frac{\omega' t}{2}\Bigg] e^{-\Gamma t}, \\
|C_2(t)|^2 &= \frac{1}{2}\Bigg[1 - \frac{2\mathcal{E}_0 \sin{(\psi-\varphi)}}{\omega'} \sin{(\omega' t)} \\
&- \frac{4\mathcal{E}_0(\omega_0-\omega)\cos{(\psi-\varphi)}}{\omega'^2} \sin^2\frac{\omega' t}{2}\Bigg] e^{-\Gamma t}.
\end{aligned}
\tag{16}
$$

The value $\psi=0$ corresponds to the radial direction.

Therefore, $|C_1(t)|^2 \neq |C_2(t)|^2 \neq 1/2$ and the spin vector does not lie in the horizontal plane. The spin oscillates in the vertical direction with the angular frequency ω'.

8 EFFECT OF RESONANCES ON SPIN MOTION IN STORAGE RINGS

The effect of resonances on spin motion in storage rings has some peculiarities. The spin-dependent part of the Hamiltonian affected by the vertical BOs can be written in the form

$$\mathcal{H} = \sigma_\rho \lambda \kappa_1 \cos(\omega t + \varphi) - \sigma_\phi \kappa_2 \sin(\omega t + \varphi),$$

where λ equals 1 and -1 for particles with negative and positive charges moving counterclockwise and clockwise, respectively,

$$\kappa_1 = \frac{1}{2} f \omega \psi_0, \quad \kappa_2 = \omega_0 \frac{\gamma - 1}{2\gamma} \psi_0,$$

and ψ_0 is the angular amplitude of the vertical BOs. The factor f is

$$f = 1 + a\gamma - \frac{1+a}{\gamma} = 1 + a\beta^2\gamma - \frac{1}{\gamma}$$

and

$$f = 1 + a\gamma$$

for electric and magnetic focusing, respectively.

For any sign of charge,

$$\omega_a = \omega + (\omega_0 - \omega)\sqrt{1 + \frac{(\kappa_2 - \kappa_1)^2}{(\omega_0 - \omega)^2}}, \quad \omega_0 \neq \omega. \tag{17}$$

It is very important that the effect of resonances is rather weak for electron, positrons, and muons ($a << 1$). It is stronger for protons ($a \sim 1$). When $a << 1$,

$$\kappa_2 - \kappa_1 = \frac{1}{2}\left[(\omega_0 - \omega)\frac{\gamma - 1}{\gamma} - \frac{\gamma^2 - 1}{\gamma}a\omega\right]\psi_0$$

and

$$\kappa_2 - \kappa_1 = \frac{1}{2}\left[(\omega_0 - \omega)\frac{\gamma - 1}{\gamma} - \frac{a\gamma^2 + 1}{\gamma}\omega\right]\psi_0$$

for electric and magnetic focusing, respectively.

In the g−2 experiment, $\dfrac{1}{\gamma^2 - 1} = a, \gamma = 29.3$ and electric focusing is used. In this case

$$\kappa_2 - \kappa_1 = \frac{1}{2}\left(\omega_0 - \omega - \frac{\omega_0}{\gamma}\right)\psi_0,$$

and the half-width of resonance curve is narrowed 29.3 times. The angular frequency of g−2 precession is given by

$$\omega_a = \omega + (\omega_0 - \omega)\sqrt{1 + \frac{1}{4}\left[1 - \frac{\omega_0}{\gamma(\omega_0 - \omega)}\right]^2 \psi_0^2}, \quad \omega_0 \neq \omega.$$

These formulae agree with the formula obtained by F. Farley for the pitch correction.

9 SUMMARY

• The exact equation of spin motion in the cylindrical coordinate system is derived. This coordinate system is convenient for describing the spin motion in storage rings. The electric dipole moments of particles and perturbations of particle orbit are taken into account.

• The formula for the frequency of g−2 precession is in the best agreement with previous results. The found formula contains the additional oscillatory term that can be used for fitting.

• The effect of resonances on spin motion in storage rings is investigated. Previous results are corrected. When $|g-2| << 1$, the effect is rather weak.