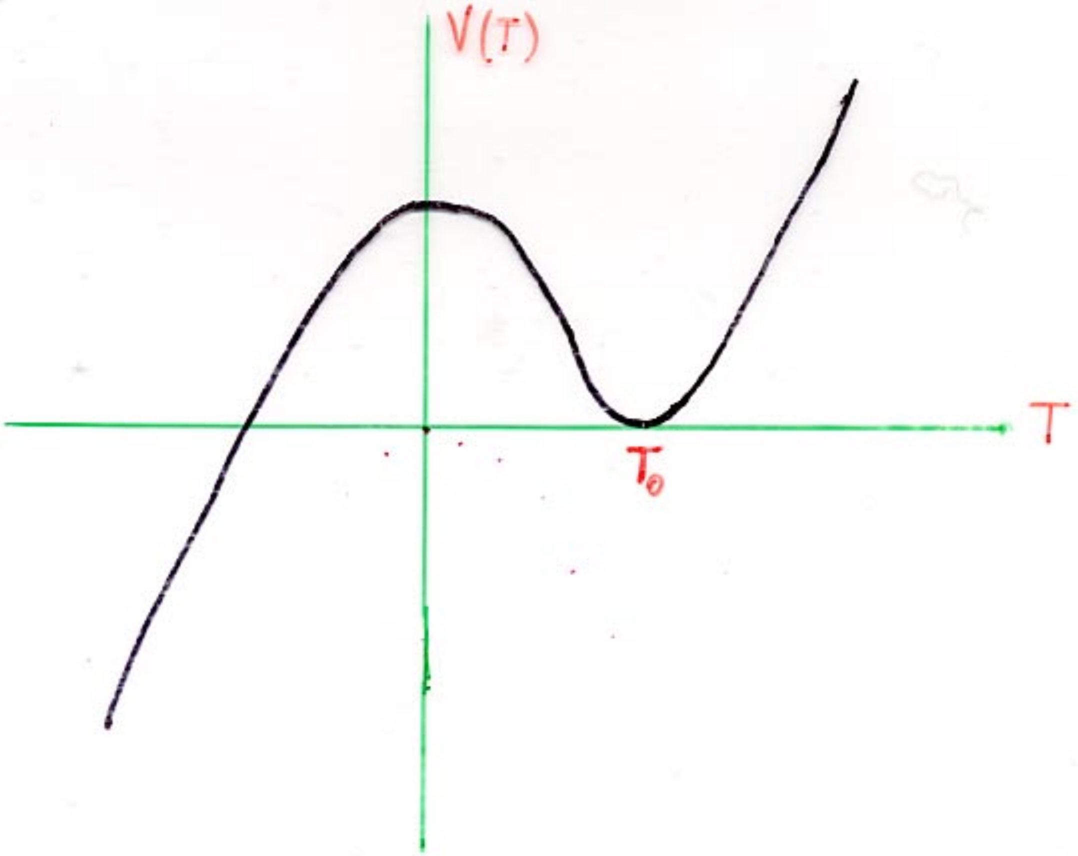
Introduction to SFT

Dubna 2003

- Apology (eulogy) of ST.
- Witten's SFT.
- Sen's conjectures and tachyon condensation.
- VSFT.
- Sliver and other solutions.
- B-field and GMS solitons.
- Split string and Moyal formulations
- The ghost sector
- Summary of VSFT
- Old and new problems
- Integrability
- Rolling tachyon

Sen's conjectures (on D=26 OBS)



$$V(T) = M\left(1 + f(T)\right) \qquad M = T_{25}$$

1) $f(T_0) = -1$

2) There exist soliton lumps that correspond to lower dimensional branes

3) The vacuum at T_0 is the closed string vacuum

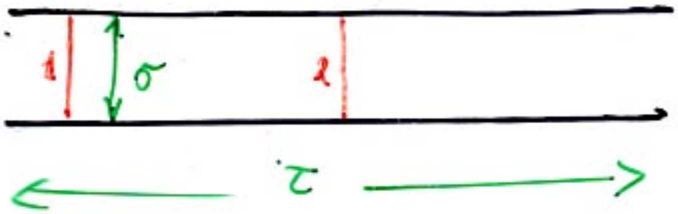
Bosonic Open String Theory

Action

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2z \, \partial X^\mu \bar{\partial} X_\mu$$

$$S_g = \frac{1}{2\pi} \int_\Sigma d^2z \, b \, \bar{\partial} c$$

World-sheet:



$0 \leq \sigma \leq \pi \qquad -\infty < \tau < +\infty$

$z = e^{t + i\sigma} \qquad t = i\tau$

Oscillator basis expansion:

$$X^\mu(z, \bar{z}) = x^\mu - i\alpha' p^\mu \ln |z|^2 + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} \left(z^{-n} + \bar{z}^{-n} \right)$$

$$c(z) = \sum_n c_n z^{-n+1} \qquad b(z) = \sum_n b_n z^{-n-2}$$

Dirac brackets:

$$[\alpha_m^\mu, \alpha_n^\nu] = m \, \delta_{m+n,0} \, \eta^{\mu\nu}$$

$$[x^\mu, p^\nu] = i \eta^{\mu\nu}$$

$$\{b_n, c_m\} = \delta_{m+n,0}$$

Classical constraints

$$L_n = 0 \qquad\qquad L_m = \frac{1}{2}\sum_k \alpha^\mu_{m-k}\,\alpha^\nu_k\,\eta_{\mu\nu}$$

become quantum constraints:

$$L_n|\phi\rangle = 0 \qquad n > 0$$

$$L_0|\phi\rangle = |\phi\rangle$$

where

$$L_n = \frac{1}{2}\sum_k :\alpha_{n-k}\cdot\alpha_k:$$

Alternatively (BRST formulation) define

$$Q_B = \sum_{n=-\infty}^{+\infty} c_n\left(L_{-n} + \frac{1}{2}L^{(gh)}_{-n}\right) - c_0$$

so that

$$Q_B = \sum_n c_n L_{-n} + \sum_{n,k}\frac{n-k}{2} :c_n c_k b_{-n-k}: - c_0$$

and impose

$$Q_B|\psi\rangle = 0 \qquad\qquad |\psi\rangle \neq Q_B|\chi\rangle$$

Vertex operators:

$$V_T = \int_{\partial\Sigma} dz\, e^{ik\cdot X}$$

$$V_A = \int_{\partial\Sigma} dz\, A_\mu \partial X^\mu e^{ikX}$$

On-shell amplitudes

$$\langle V_1 \cdots V_N \rangle \qquad k_i^2 = -M_i^2$$

can be computed $\left(\Rightarrow \text{EOM} \Leftrightarrow \text{LEEA} \right)$
(to some extent)

In general we need off-shell information
(one-loop, ...)

$$\Downarrow$$

String Field Theory

$$\Downarrow$$

String Horography

Bosonic Open String Field Theory (D=26)

Action

$$S = -\frac{1}{g_o^2}\left(\frac{1}{2}\int \Psi * Q_B \Psi + \frac{1}{3}\int \Psi * \Psi * \Psi\right)$$

where

$$Q_B^2 = 0$$

$$\int Q_B \Psi = 0$$

$$(A * B) * C = A * (B * C)$$

$$Q_B(A * B) = (Q_B A) * B + (-1)^{|A|} A * (Q_B B)$$

Gauge invariance:

$$\delta\Psi = Q_B \Lambda + \Psi * \Lambda - \Lambda * \Psi$$

By definition $\quad |A| =$ Grassmannality of A

$$\#_g(\Psi) = \#_g(Q_B) = 1$$

$$\#_g(\Lambda) = 0$$

$$\#_g(*) = 0$$

$$\#_g\left(\int\right) = -3$$

Definitions:

1) The vacuum (SL(2, R) invariant)

$$\alpha_n^\mu |0\rangle = 0 \qquad n \geq 0$$

$$c_n |0\rangle = 0 \qquad n > 1$$

$$b_n |0\rangle = 0 \qquad n \geq -1$$

2) The string Field

$$\Psi[x(\sigma)]$$

or

$$|\Psi\rangle = \left(\phi(x) + A_\mu(x)\,\alpha_{-1}^\mu + B_{\mu\nu}(x)\,\alpha_{-1}^\mu \alpha_{-1}^\nu + \ldots\right) c_1 |0\rangle$$

Relation between the two: define

$$a_n^\mu = \frac{1}{\sqrt{n}}\alpha_n^\mu \qquad\qquad a_n^{\mu\dagger} = \frac{1}{\sqrt{n}}\alpha_{-n}^\mu$$

$$\hat{x}_n = \frac{i}{\sqrt{2n}}\left(a_n - a_n^\dagger\right) \qquad\qquad \hat{p}_n = \sqrt{\frac{n}{2}}\left(a_n + a_n^\dagger\right)$$

$$\hat{x}(\sigma) = \hat{x}(\sigma, \tau = 0) = \hat{x}_0 + \sqrt{2}\sum_{n=1}^{\infty}\hat{x}_n \cos n\sigma$$

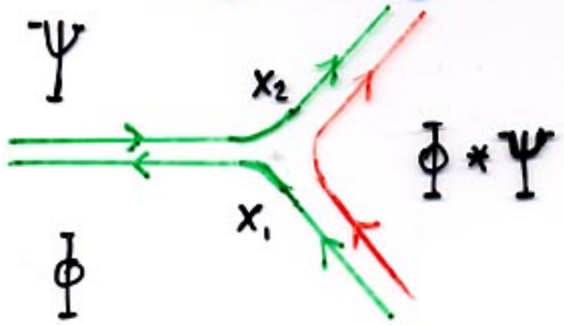
then

$$\Psi[x(\sigma)] = \langle \hat{x}(\sigma) | \Psi \rangle$$

$$|\hat{x}(\sigma)\rangle = \exp \sum_{n=1}^{\infty}\left(-\frac{1}{2} n\, x_n x_n - x_0^2 - i\sqrt{2n}\, a_n^\dagger x_n - 2i\, a_0^\dagger x_0 + \frac{1}{2} a_n^\dagger a_n^\dagger\right)|0\rangle$$

3) The $*$ product.

Star product of $\Phi[x_1]$ with $\Psi[x_2]$ means identifying R half of x_1 with L half of x_2 and integrating over



- First formulation (functional)

$$(\Phi * \Psi)[z(\sigma)] = \int \Phi[x(\sigma)]\, \Psi[y(\sigma)] \prod_{\frac{\pi}{2} \le \sigma \le \pi} \delta[x(\sigma) - y(\pi-\sigma)] \prod_{\frac{\pi}{2} \le \sigma \le \pi} dx(\sigma) \prod_{0 \le \sigma \le \frac{\pi}{2}} dy(\sigma)$$

$$z(\sigma) = x(\sigma) \qquad 0 \le \sigma \le \frac{\pi}{2}$$

$$z(\sigma) = y(\sigma) \qquad \frac{\pi}{2} \le \sigma \le \pi$$

- Second formulation (operator)

3-string vertex $\langle V_3|$

$$\boxed{\begin{aligned}\langle V_3| \equiv {}_1\langle 0| c_{-1}^{(1)} c_0^{(1)} \otimes {}_2\langle 0| c_{-1}^{(2)} c_0^{(2)} \otimes {}_3\langle 0| c_{-1}^{(3)} c_0^{(3)} \cdot \int dp_1\, dp_2\, dp_3\, \delta(p_1+p_2+p_3) \\ \cdot \exp\left[-\left(\frac{1}{2}\sum_{r,s=1}^{3} \sum_{n,m=0}^{\infty} \eta_{\mu\nu}\, a_n^{(r)\mu} V_{nm}^{rs} a_m^{(s)\nu} + \sum_{r,s=1}^{3} \sum_{\substack{n\ge 1\\ m \ge 0}} c_n^{(r)} \tilde{V}_{nm}^{rs} b_m^{(s)}\right)\right]\end{aligned}}$$

where $\left[a_m^{(r)\mu}, a_n^{(s)\nu\dagger}\right] = \eta^{\mu\nu}\delta_{mn}\delta^{rs}$ $\qquad a_n = \frac{\alpha_n}{\sqrt{n}}$

$$\hat{p}|p\rangle = p|p\rangle \quad , \quad \langle p|p'\rangle = \delta(p+p')$$

Then

$$\boxed{{}_3\langle\phi * \psi| = \langle V_3|\phi\rangle_1|\psi\rangle_2}$$

where

$$\langle\phi| = bpz(|\phi\rangle)$$

Rules for bpz:

$$bpz(\alpha_{-m}^{\mu}) = -(-1)^m \alpha_m^{\mu}$$

$$bpz(c_{-m}) = -(-1)^m c_m$$

$$bpz(b_{-m}) = (-1)^m b_m$$

Use:

$$\langle 0|\, e^{\lambda_i a_i - \frac{1}{2} a_i P_{ij} a_j} \; e^{\mu_i a_i^\dagger - \frac{1}{2} a_i^\dagger Q_{ij} a_j^\dagger}|0\rangle =$$

$$= (\det K)^{-1/2}\, e^{\mu^T K^{-1}\lambda - \frac{1}{2}\lambda^T Q K^{-1}\lambda - \frac{1}{2}\mu^T K^{-1} P \mu}$$

with $\quad K = 1 - PQ$

Neumann coefficients

$$\left(\frac{1+ix}{1-ix}\right)^{1/3} = \sum_{n\ \text{even}} A_n x^n + i \sum_{n\ \text{odd}} A_n x^n$$

$$\left(\frac{1+ix}{1-ix}\right)^{2/3} = \sum_{n\ \text{even}} B_n x^n + i \sum_{n\ \text{odd}} B_n x^n$$

$$N^{r,\pm r}_{nm} = \begin{cases} \frac{1}{3(n\pm m)} (-1)^n (A_n B_m \pm B_n A_m) & m+n \text{ even} \quad n\neq m \\ 0 & m+n \text{ odd} \end{cases}$$

$$N^{r,\pm(r+1)}_{nm} = \begin{cases} \frac{1}{6(n\pm m)} (-1)^{n+1} (A_n B_m \pm B_n A_m) & m+n \text{ even} \quad n\neq m \\ \frac{1}{6(n\pm m)} \sqrt{3}\, (A_n B_m \mp B_n A_m) & m+n \text{ odd} \end{cases}$$

$$N^{r,\pm(r-1)}_{nm} = \begin{cases} \frac{1}{6(n\pm m)} (-1)^{n+1} (A_n B_m \mp B_n A_m) & m+n \text{ even} \quad m\neq n \\ -\frac{1}{6(n\mp m)} \sqrt{3}\, (A_n B_m \pm B_n A_m) & m+n \text{ odd} \end{cases}$$

$$V^{rs}_{nm} = -\sqrt{nm}\left(N^{rs}_{nm} + N^{r,-s}_{nm}\right) \qquad m\neq n, \quad m,n\neq 0$$

$$V^{rr}_{nn} = -\frac{1}{3}\left(2\sum_{k=0}^{n} (-1)^{n-k} A_k^2 - (-1)^n - A_n^2\right) \qquad n\neq 0$$

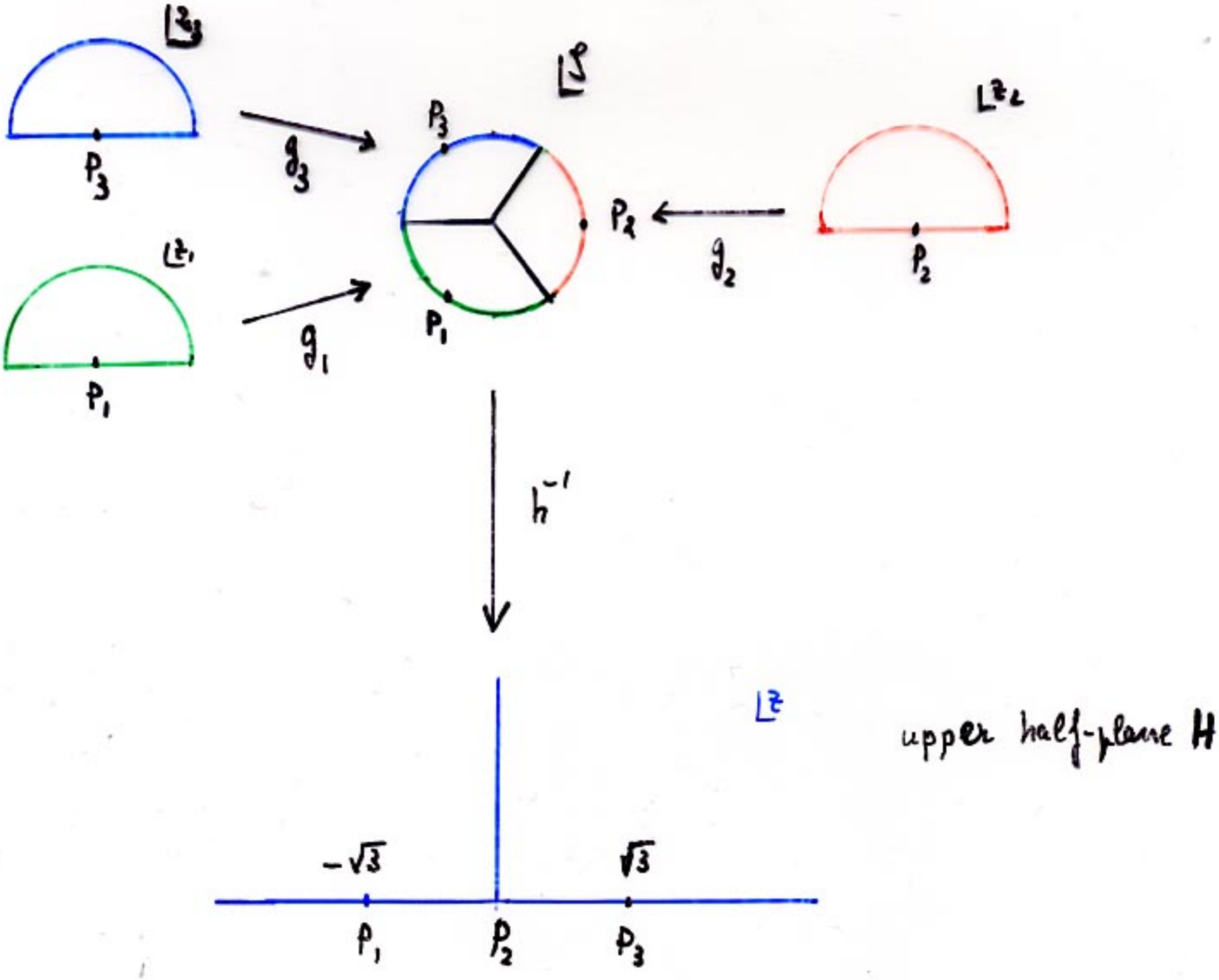
$$V^{r\,r+1}_{nn} = V^{r\,r+2}_{nn} = \frac{1}{2}\left((-1)^n - V^{rr}_{nn}\right) \qquad n\neq 0$$

$$V^{rs}_{0n} = -\sqrt{2n}\left(N^{rs}_{0n} + N^{r,-s}_{0n}\right) \qquad n\neq 0$$

$$V^{rr}_{00} = \ln\frac{27}{16}$$

Third formulation

CFT formulation



$$g_k(z_k) = e^{\frac{2\pi i}{3}(k-2)} \left(\frac{1+i z_k}{1-i z_k}\right)^{2/3} \qquad z = h^{-1}(\xi) = -i\,\frac{\xi-1}{\xi+1}$$

$$f_k(z_k) = h^{-1}\circ g_k(z_k)$$

Then

$$\int \Phi * \Phi * \Phi = \langle f_1\circ\Phi(0)\ f_2\circ\Phi(0)\ f_3\circ\Phi(0)\rangle_H$$

Matter Neumann coefficients

$$N^{rs}_{nm} = \langle V_{123} | \alpha^{(r)}_{-n} \alpha^{(s)}_{-m} | 0 \rangle_{123} = \langle f_r[\alpha_{-n}] \, f_s[\alpha_{-m}] \rangle =$$

$$= -\frac{1}{nm} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} f_r'(z) \frac{1}{(f_r(z) - f_s(w))^2} f_s'(w)$$

so that

$$V^{rs}_{nm} = (-1)^{n+m} \sqrt{nm} \, N^{rs}_{nm}$$

Decomposition

$$N^{rs}_{nm} = \frac{1}{3\sqrt{nm}} \left(E_{nm} + \bar{\alpha}^{r-s} U_{nm} + \alpha^{r-s} \bar{U}_{nm} \right) \qquad \alpha = e^{\frac{2\pi i}{3}}$$

where

$$E_{nm} = \frac{-1}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \left(\frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2} \right) = (-1)^n \delta_{nm}$$

$$U_{nm} = \frac{-1}{3\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \left(\frac{f^2(w)}{f^2(z)} + 2\frac{f(z)}{f(w)} \right) \left(\frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2} \right)$$

$$\bar{U}_{nm} = (-1)^{n+m} U_{nm}$$

Properties

$$N^{rs}_{nm} = N^{sr}_{mn}$$

$$N^{rs}_{mn} = (-1)^{n+m} N^{sr}_{mn}$$

$$N^{rs}_{nm} = N^{r+1,s+1}_{nm}$$

Basic property

$$\sum_{k=1}^{\infty} U_{nk} U_{km} = \delta_{nm}$$

it follows

$$X^{rs} = C V^{rs} \qquad \left[X^{rs}, X^{r's'}\right] = 0 \quad \forall\, r,s,r',s'$$

Zero modes

$$N_{0n}^{rs} = -\frac{1}{n}\oint \frac{dz}{2\pi i}\,\frac{1}{z^n}\, f_s'(z)\,\frac{1}{f_r(0)-f_s(z)} = \frac{1}{3}\left(E_n + \bar{\alpha}^{r-s} U_n + \alpha^{r-s}\bar{U}_n\right)$$

where

$$E_n = -\frac{4i}{n}\oint \frac{dz}{2\pi i}\,\frac{1}{z^n}\,\frac{1}{1+z^2}\,\frac{f^3(z)}{1-f^3(z)} = 2\,\frac{i^n}{n}$$

$$U_n = -\frac{4i}{n}\oint \frac{dz}{2\pi i}\,\frac{1}{z^n}\,\frac{1}{1+z^2}\,\frac{f^2(z)}{1-f^3(z)} = \frac{a_n}{n}$$

$$\bar{U}_n = (-1)^n \frac{a_n}{n}$$

$$\left(\frac{1+iz}{1-iz}\right)^{1/3} = \sum_{n=0}^{\infty} a_n z^n$$

Imposing the gauge fixing condition

$$\sum_{r=1}^{3} N_{0n}^{rs} = 0 \;\Rightarrow\; \hat{N}_{0n}^{rs} = N_{0n}^{rs} - \frac{1}{3}E_n$$

$$V_{0n}^{rs} = -\sqrt{2n}\,\hat{N}_{0n}^{rs}$$

4) The BRST charge

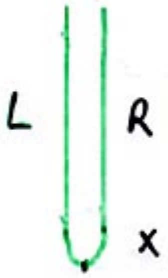
$$\boxed{Q_B = \sum_{n=-\infty}^{+\infty} c_n L_{-n}^{(m)} + \sum_{m,k} \frac{m-k}{2} : c_m c_k b_{-m-k} : - c_0}$$

$Q_B^2 = 0$ in $D = 26$

$\{Q_B , b_0\} = L_0^{tot}$ $\rightarrow$ Siegel gauge $b_0|\psi\rangle = 0$

5) Integration

Integration corresponds to identifying L and R of string and integrating over



$\Leftrightarrow \quad \int \Phi[x] = \langle I | \Phi \rangle$

where

$$I[x(\sigma)] = \langle x(\sigma) | I \rangle = \prod_{0 \le \sigma \le \pi/2} \delta\big(x(\sigma) - x(\pi - \sigma)\big)$$

More explicitly

$$\int \Phi \equiv \int \mathcal{D}x(\sigma) \prod_{0 \le \sigma \le \frac{\pi}{2}} \delta\big(x(\sigma) - x(\pi - \sigma)\big)\, \Phi[x(\sigma)]$$

In operator language $\langle I | = \langle I_m | \otimes \langle I_g |$:

$$\langle I_m | = \langle 0 | e^{-\frac{1}{2}\sum_n a_n C_{nm} a_m}$$

$C_{nm} = (-1)^n \delta_{nm}$

$$\langle I_g | = \langle 0 | e^{-\sum_{n=1}^{\infty} (-1)^n c_n b_n}$$

Some examples

- $|I\rangle$ is the identity for the $*$ product

$$\left(\Phi * I\right)[z(\sigma)] = \int \Phi[x(\sigma)] \prod_{0\le\sigma\le\frac{\pi}{2}} \delta\left(y(\sigma)-y(\pi-\sigma)\right) \cdot$$

$$\cdot \prod_{\frac{\pi}{2}\le\sigma\le\pi} \delta\left[x(\sigma)-y(\pi-\sigma)\right] \prod_{\frac{\pi}{2}\le\sigma\le\pi} dx(\sigma) \prod_{0\le\sigma\le\frac{\pi}{2}} dy(\sigma)$$

$$= \int \Phi[x(\sigma)] \prod_{\frac{\pi}{2}\le\sigma\le\pi} \delta\left[x(\sigma)-y(\sigma)\right] \prod_{\frac{\pi}{2}\le\sigma\le\pi} dx(\sigma)$$

$$= \Phi[y(\sigma)] \quad \frac{\pi}{2}\le\sigma\le\pi \qquad = \Phi[x(\sigma)] \quad 0\le\sigma\le\frac{\pi}{2}$$

$$= \Phi[z(\sigma)]$$

Another representation of $|I\rangle$:

$$|I\rangle = e^{L_{-2}-\frac{1}{2}L_{-4}+\frac{1}{2}L_{-6}-\frac{7}{12}L_{-8}\cdots}|0\rangle$$

If we restrict Φ to

$$|\Phi\rangle = \int d^d k \left(\phi(k) + A_\mu(k)\, \alpha_{-1}^\mu \right) c_1 |k\rangle$$

the action becomes (Siegel gauge $b_0|\Phi\rangle = 0$)

$$S = \frac{1}{g_0^2} \int d^d x \Bigg(-\frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi + \frac{1}{2\alpha'} \phi^2 - \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu$$

$$- \frac{1}{3} \left(\frac{3\sqrt{3}}{4} \right)^3 \tilde{\phi}^3(x) - \frac{3\sqrt{3}}{4}\, \tilde{\phi} \tilde{A}_\mu \tilde{A}^\mu +$$

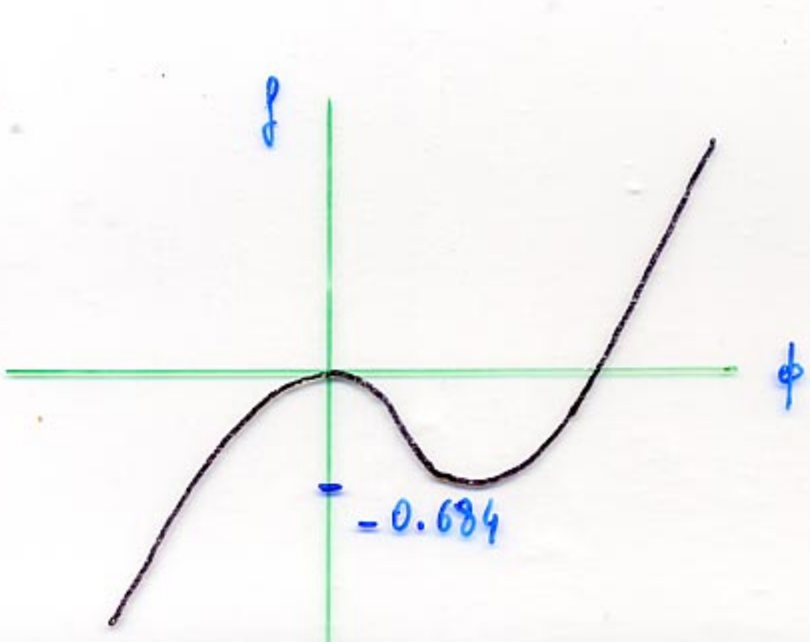
$$- \frac{3\sqrt{3}}{8} \alpha' \left(\partial_\mu \partial_\nu \tilde{\phi}\, \tilde{A}^\mu \tilde{A}^\nu + \tilde{\phi}\, \partial_\mu \tilde{A}^\nu \partial_\nu \tilde{A}^\mu - 2\, \partial_\mu \tilde{\phi}\, \partial_\nu \tilde{A}^\mu \tilde{A}^\nu \right) \Bigg)$$

where

$$\tilde{f}(x) = e^{-\alpha' \ln \frac{4}{3\sqrt{3}} \partial_\mu \partial^\mu} f(x)$$

Considering only the tachyon and dropping derivatives

$$S \longrightarrow \frac{1}{g_0^2} \int d^d x \left(\frac{1}{2\alpha'} \phi^2 - \frac{1}{3} \left(\frac{3\sqrt{3}}{4} \right)^3 \phi^3 \right) \equiv - \frac{f(\phi) V}{2\pi^2 \alpha'^3}$$



Level truncation

level	$f(T_0)$
(0,0)	-0.684
(2,4)	-0.949
(2,6)	-0.959
(4,8)	-0.986
(4,12)	-0.988
(6,12)	-0.99514
(6,18)	-0.99518
(8,16)	-0.99777
(8,20)	-0.99793
(10,20)	-0.99912

Level (2.6)

$$|T\rangle = \left(\phi\, c_1 - \beta_1\, c_{-1} + \frac{v}{\sqrt{13}} L^{(m)}_{-2} c_1\right)|0\rangle$$

gives

$$f(T) = 2\pi^2 \alpha'^3 \Big(-\frac{1}{2\alpha'}\phi^2 + \frac{3^3\sqrt{3}}{2^6}\phi^3 - \frac{1}{2\alpha'}\beta_1^2 + \frac{1}{2\alpha'} v^2$$
$$- \frac{11\cdot 3\sqrt{3}}{2^6}\phi^2\beta_1 - \frac{5\cdot 3\sqrt{39}}{2^6}\phi^2 v + \frac{19\sqrt{3}}{3\cdot 2^6}\phi\beta_1^2$$
$$+ \frac{581\sqrt{3}}{3^2\cdot 2^6}\phi v^2 + \frac{5\cdot 11\sqrt{39}}{3^2\cdot 2^5}\phi\beta_1 v - \frac{1}{2^6\sqrt{3}}\beta_1^3$$
$$- \frac{5\cdot 19\sqrt{39}}{2^6\cdot 3^4} v\beta_1^2 - \frac{6391\sqrt{3}}{2^6\cdot 3^5} v^2\beta_1 - \frac{20951\sqrt{39}}{2^6\cdot 3^5\cdot 13} v^3 \Big)$$

New vacuum has

- no tachyon
- no massless vector field
- $\frac{1}{g_{eff}} = \frac{V(T)}{g_0}$
- . . . . .

Vacuum String Field Theory

Defines a SFT corresponding to closed string vacuum. Just shift

$$\Phi = \Phi_0 + \tilde{\Phi}$$

Φ_0 corresponds to T_0

Then

$$S(\Phi_0+\tilde{\Phi}) = -V_{25}T_{25} - \frac{1}{g_0^2}\int\Big[\frac{1}{2}(\Phi_0+\tilde{\Phi}) * Q(\Phi_0+\tilde{\Phi}) + \frac{1}{3}(\Phi_0+\tilde{\Phi}) * (\Phi_0+\tilde{\Phi}) * (\Phi_0+\tilde{\Phi})\Big]$$

$$= -\frac{1}{g_0^2}\int\Big[\frac{1}{2}\tilde{\Phi} * \mathcal{Q}\tilde{\Phi} + \frac{1}{3}\tilde{\Phi} * \tilde{\Phi} * \tilde{\Phi}\Big]$$

where

$$\mathcal{Q}\tilde{\Phi} = Q_B\tilde{\Phi} + \frac{1}{2}\left(\Phi_0 * \tilde{\Phi} + \tilde{\Phi} * \Phi_0\right)$$

Possible field redefinition

$$\tilde{\Phi} = e^K \Psi$$

Summing up <u>we postulate</u> at the closed string vacuum

$$S = -\frac{1}{g_0^2}\int\Big[\frac{1}{2}\Psi * \mathcal{Q}\Psi + \frac{1}{3}\Psi * \Psi * \Psi\Big]$$

The new BRST charge $\mathcal{Q}$ satisfies

$$\mathcal{Q}^2 = 0 \qquad \mathcal{Q}(\psi * \chi) = \mathcal{Q}\psi * \chi + (-1)^{\psi}\psi * (\mathcal{Q}\chi)$$

The new BRST charge must satisfy

$$\mathcal{Q}^2 = 0$$

$$\mathcal{Q}(A * B) = (\mathcal{Q}A) * B + (-1)^{|A|} A * (\mathcal{Q}B)$$

$$\langle \mathcal{Q}A, B \rangle = -(-1)^{|A|} A * (\mathcal{Q}B)$$

and

- $\mathcal{Q}$ must have vanishing cohomology
  (no open string states)

- $\mathcal{Q}$ must be universal
  (no dependence on BCFT)

Examples of $\mathcal{Q}$'s :

- $\mathcal{Q} = c_0$

- $\mathcal{Q} \equiv \mathcal{C}_n = c_n + (-1)^n c_{-n}$ $\qquad n = 0, 1, 2, \ldots$

- $\mathcal{Q} \equiv \sum_{n=0}^{\infty} a_n \mathcal{C}_n$

Proof: define $\mathcal{B}_n = \frac{1}{2}\left(b_n + (-1)^n b_{-n}\right) \longrightarrow \{\mathcal{C}_n, \mathcal{B}_n\} = 1$

Therefore, if $\mathcal{C}_n \psi = 0 \longrightarrow \psi = \mathcal{C}_n(\mathcal{B}_n \psi) = \{\mathcal{C}_n, \mathcal{B}_n\}\psi$

Now search for classical solution of EOM of VSFT

$$\mathcal{Q}\Psi = -\Psi * \Psi$$

Ansatz

$$\Psi = \Psi_m * \Psi_g$$

So EOM splits

$$\mathcal{Q}\Psi_g = -\Psi_g *^g \Psi_g \qquad \Psi_m = \Psi_m *^m \Psi_m$$

and

$$S|_\Psi = -\frac{1}{6g_0^2} \langle \Psi_g | \mathcal{Q} \Psi_g \rangle_g \langle \Psi_m | \Psi_m \rangle_m \equiv K \langle \Psi_m | \Psi_m \rangle_m$$

Method of Kostelecky-Potting

Three string vertex $|V_3\rangle$:

$$|V_3\rangle = \int d^{26}p_{(1)}\, d^{26}p_{(2)}\, d^{26}p_{(3)}\, \delta^{(26)}\left(p_{(1)} + p_{(2)} + p_{(3)}\right) e^{-E} |0,p\rangle_{1,2,3}$$

with

$$E = \frac{1}{2} \sum_{\substack{r,s=1 \\ m,n\geq 1}}^{3} \eta_{\mu\nu}\, a_m^{(r)\mu\dagger} V_{mn}^{rs} a_n^{(s)\nu\dagger} + \sum_{\substack{r,s=1 \\ n\geq 1}}^{3} \eta_{\mu\nu}\, p_{(r)}^{\mu} V_{0n}^{rs} a_n^{(s)\nu\dagger} + \frac{1}{2}\sum_{r=1}^{3} \eta_{\mu\nu}\, p_{(r)}^{\mu} V_{00}^{rr} p_{(r)}^{\nu}.$$

and

$$|0,p\rangle_{123} = |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$$

For space-time translational invariant solutions

$$E = \frac{1}{2} \sum_{\substack{r,s=1 \\ m,n\geq 1}}^{3} \eta_{\mu\nu}\, a_m^{(r)\mu\dagger} V_{mn}^{rs} a_n^{(s)\nu\dagger}$$

ansatz:

$$|\Psi_{tw}\rangle = \mathcal{N}^{26}\, e^{-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn} a_m^{\mu\dagger} a_n^{\nu\dagger}} |0\rangle$$

Now impose

$$|\Psi_m *^m \Psi_m\rangle_3 \equiv {}_1\langle\Psi_m|\,{}_2\langle\Psi_m|V_3\rangle = |\Psi_m\rangle_3$$

Get equation

$$|\Psi_m * \Psi_m\rangle_3 = \mathcal{N}^{52} \det\left[(1-\Sigma\mathcal{V})^{-1/2}\right]^{26} \cdot$$

$$\cdot \exp\left[-\frac{1}{2}\eta_{\mu\nu}\left\{\chi^{\mu T}\frac{1}{1-\Sigma\mathcal{V}}\Sigma\chi^{\nu} + a^{(3)\mu+}\cdot V^{33}\cdot a^{(3)\nu+}\right\}\right]|0\rangle_3$$

where

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}$$

$$\chi^{\mu T} = \left(a^{(3)\mu+}V^{31},\ a^{(3)\mu+}V^{32}\right) \qquad \chi^{\mu} = \begin{pmatrix} V^{13}a^{(3)\mu+} \\ V^{23}a^{(3)\mu+} \end{pmatrix}$$

Equating and using $V^{r+1,s+1} = V^{r,s}$ mod 3

$$(*) \qquad S = V^{11} + \left(V^{12}, V^{21}\right)\frac{1}{1-\Sigma\mathcal{V}}\Sigma\begin{pmatrix} V^{21} \\ V^{12} \end{pmatrix}$$

Solve for S. seems hopeless

But . . . define

$$X^{rs} = C V^{rs}$$

$$C_{nm} = (-1)^m \delta_{nm}$$

$$[X^{rs}, X^{r's'}] = 0$$

Set

$$X = X^{11} \qquad\qquad T = CS$$

then (*) becomes

$$(T-1)\left(XT^2 - (1+X)T + X\right) = 0$$

i.e.

$$S = CT \qquad T = \frac{1}{2X}\left(1 + X - \sqrt{(1+3X)(1-X)}\right)$$

Finally the solution is

$$\boxed{|\Psi_m\rangle = \left(\det(1-X)\det(1+T)\right)^{13} e^{-\frac{1}{2}\sum_{m,n\geq 1} a_m^{+} S_{mn} a_n^{+}}|0\rangle}$$

and

$$\boxed{S|_\Psi = K\,\frac{V^{(26)}}{(2\pi)^{26}}\left(\det(1-X)^{3/4}\det(1+3X)^{1/4}\right)^{26}}$$

$|\Psi_m\rangle$ is identified with the D25-brane.

Lump solutions

They are supposed to represent D-(25-k)-branes.

k transverse directions, $\alpha = 1, \ldots, k$.

Replace

$$|p^\alpha\rangle = \frac{1}{\pi^{k/4}} \, e^{-\frac{1}{2}p^\alpha p^\alpha + \sqrt{2}\, a_0^{\alpha+} p^\alpha - \frac{1}{2} a_0^{\alpha+} a_0^{\alpha+}} |\Omega\rangle$$

where

$$a_0^\alpha = \frac{1}{\sqrt{2}}\left(\hat{p}^\alpha - i\hat{x}^\alpha\right) \qquad a_0^{\alpha+} = \frac{1}{\sqrt{2}}\left(\hat{p}^\alpha + i\hat{x}^\alpha\right)$$

$$[a_0^\alpha, a_0^{\beta+}] = \delta^{\alpha\beta}$$

Integrate over p^α. The relevant vertex is:

$$|V_3\rangle = \exp\left(-\frac{1}{2}\sum_{\substack{r,s\\ m,n\geq 1}} \eta_{\mu\nu}\, a_m^{(r)\mu+}\, V_{mn}^{rs}\, a_n^{(s)\nu+}\right)|0,p\rangle_{123} \cdot$$

$$\cdot \left(\frac{\sqrt{3}}{(16\pi)^{1/4}}\left(V_{00}^{rs}+1\right)\right)^{-k} \exp\left(-\frac{1}{2}\sum_{\substack{r,s\\ M,N\geq 0}} a_M^{(r)\alpha+}\, V'^{rs}_{MN}\, a_N^{(s)\alpha+}\right)|\Omega\rangle$$

$\mu = 0, \ldots, 25-k-1$ $\qquad$ $M = \{0, m\}$

The solution of $|\Psi_m\rangle * |\Psi_m\rangle = |\Psi_m\rangle$ is

$$|\Psi'_m\rangle = \left(\sqrt{\det(1-X)\det(1+T)}\right)^{26-k} e^{-\frac{1}{2}\sum_{m,n\geq 1} a_m^\dagger S_{mn} a_n^\dagger} |0\rangle$$

$$\otimes \left(\frac{\sqrt{3}}{(16\pi)^{1/4}}\left(V_{00}^{rr}+1\right)\right)^k \left(\det(1-X')\det(1+T')\right)^{k/2} e^{-\frac{1}{2}\sum_{\substack{M,N\geq 0}} a_M^{\alpha\dagger} S_{MN} a_N^{\alpha\dagger}} |\Omega\rangle$$

Gives the action

$$S_{\Psi'} = K \frac{V^{(26-k)}}{(2\pi)^{26-k}} \left(\det(1-X)^{3/4}\det(1+3X)^{1/4}\right)^{26-k} \cdot$$

$$\cdot \left(\frac{3}{(16\pi)^{1/2}}\left(V_{00}^{rr}+1\right)^2\right)^k \left(\det(1-X')^{3/4}\det(1+3X')^{1/4}\right)^k$$

Ratio of tensions:

$$\frac{T_{24-k}}{2\pi\alpha' T_{25-k}} = \frac{3}{\sqrt{16\pi}}\left(V_{00}^{rr}+1\right)^2 \frac{\det(1-X')^{3/4}\det(1+3X')^{1/4}}{\det(1-X)^{3/4}\det(1+3X)^{1/4}}$$

Numerically this $= 1$. (Okuyama)

Moyal product in $\mathbb{R}^d$ $\theta^{\mu\nu} = -\theta^{\nu\mu}$

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)\, g(y)\Big|_{y=x}$$

In particular

$$x^\mu * x^\nu - x^\nu * x^\mu = i\,\theta^{\mu\nu}$$

$$e^{ipx} * e^{iqx} = e^{i(p+q)x}\; e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu}$$

Moyal product defines a n.c. associative algebra $\mathcal{A}_\theta$.

$$\int d^d x\; f * g = \int d^d x\; f g$$

GFT in n.c. $\mathbb{R}^d$

$$\delta_\lambda A_\mu = \partial_\mu \lambda + i \lambda * A_\mu - i A_\mu * \lambda$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i A_\nu * A_\mu - i A_\mu * A_\nu$$

Action

$$S = -\frac{1}{4g^2}\int d^D x\; \mathrm{Tr}\left(F * F\right)$$

if $A_\mu = A_\mu^a t^a$, $(t^a)^+ = t^a$

Simple example in **2+1 D** and $\theta \to \infty$.
Coordinates : x^1, x^2, t with $z = x^1 + i x^2$
Rescale $x^i \to x^i \sqrt{\theta}$, then

$$E = \frac{1}{g^2} \int d^2 z \left(\frac{1}{2} (\partial \phi)^2 + \theta V_*(\phi) \right)$$

In the limit $\theta \to \infty$

$$E = \frac{\theta}{g^2} \int d^2 z \; V(\phi)$$

Extremum

$$\frac{\partial V}{\partial \phi} = 0$$

Example (cubic potential):

$$m^2 \phi + b_3 \; \phi * \phi = 0$$

i.e.

$$\boxed{\phi_0 * \phi_0 = \phi_0}$$

Solution

$$\phi_0(x) = 2 e^{-r^2}, \qquad r^2 = x_1^2 + x_1^2$$

Rescaling back

$$\phi_0(x) = 2 e^{-\frac{r^2}{\theta}}$$

Noncommutative Solitons

Two noncommutative coordinates

$$[x^1, x^2]_* = i\theta$$

can be mimicked by two quantum operators $\hat{p}, \hat{q}$:

$$[\hat{q}, \hat{p}] = i$$

Then use Weyl quantization:

there is a 1-1 correspondence between the algebra of function with $*$ product and the algebra of operators in Hilbert space

Correspondence: $p, q \longleftrightarrow \hat{p}, \hat{q}$

For any classical function $f(p,q)$ introduce the Fourier transform

$$\hat{f}(k_q, k_p) = \int dp\, dq\; e^{i(k_q q + k_p p)} f(p,q)$$

and operator

$$U(k_q, k_p) = e^{-i(k_q \hat{q} + k_p \hat{p})}$$

The correspondence between classical functions $f(p,q)$ and quantum operators is given by:

$$f(q,p) \longleftrightarrow \hat{O}_f$$

$$\hat{O}_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int dk_p\, dk_q\ U(k_q,k_p)\ \tilde{f}(k_q,k_p)$$

$$f(q,p) = \int dk_p\ e^{-ipk_p}\ \langle q+\tfrac{k_p}{2}|\,\hat{O}_f(\hat{q},\hat{p})\,|q-\tfrac{k_p}{2}\rangle$$

Examples:

$$\int dq\, dp\, f(q,p) = 2\pi\, \mathrm{Tr}_{\mathcal{H}}\, \hat{O}_f = 2\pi \int dq\ \langle q|\,\hat{O}_f\,|q\rangle$$

$$\boxed{\hat{O}_f\,\hat{O}_g = \frac{1}{(2\pi)^2}\int dk_p\, dk_q\ U(k_q,k_p)\ \widetilde{f*g}\,(k_q,k_p) = \hat{O}_{f*g}}$$

$$[\hat{O}_f,\hat{O}_g] = \hat{O}_{f*g-g*f}$$

Consider the previous example $(\theta \to \infty)$

$$S = \int dt\, dx^1 dx^2 \quad V_*(\phi)$$

Now, Weyl-transform :

$$\phi \to \hat{O}_\phi \equiv \hat{\phi} \qquad\qquad S = 2\pi\theta \int dt\, \mathrm{Tr}_{\mathcal{H}}\, V(\hat{\phi})$$

The eq. of motion is : $V'(\phi) = 0$

$$V'(\phi) = \text{const}\ \ \phi(\phi-\lambda_1)\cdots(\phi-\lambda_{n-1}) = 0$$

Now, if $\hat{P}$ is a projector, the configuration

$$\hat{\phi} = \lambda_i \hat{P} \qquad\qquad \hat{P}^2 = \hat{P}$$

is a solution, since $\qquad E = 2\pi\theta\, V(\lambda_i)\, \mathrm{Tr}_{\mathcal{H}} \hat{P}$

$$\hat{P}(1-\hat{P}) = 0$$

In general

$$\hat{\phi} = \sum_i \lambda_i \hat{P}_i \qquad\qquad \hat{P}_i \perp \hat{P}_j$$

is a non-trivial solution.

Let $a = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}$, $[a, a^+] = 1$ and let

$|n\rangle$ be a basis of harmonic oscillator eigenstates.

Consider the operator $|n\rangle\langle m|$ and its Weyl transform

$$f_{n,m}(q,p) = \int dy\, e^{-ipy} \langle q + \tfrac{y}{2} | n \rangle \langle m | q - \tfrac{y}{2} \rangle$$

Adapting to $(q, p) = (x^1, x^2)$ one finds

$$f_{n,m}(r,\phi) = 2 e^{-r^2} \sqrt{\frac{n!}{m!}} (-1)^n (2r^2)^{\frac{m-n}{2}} e^{i\phi(m-n)} L_n^{m-n}(2r^2)$$

In particular

$$f_{0,0}(x_1, x_2) = 2 e^{-(x_1^2 + x_2^2)}$$

This corresponds to the projector

$$\hat{P} = |0\rangle\langle 0|$$

The energy of the corresponding solution is:

$$E = 2\pi\theta \, \mathrm{Tr}_{\mathcal{H}} V(\lambda_i \hat{P}) = 2\pi\theta V(\lambda_i) \Longleftarrow \mathrm{Tr}_{\mathcal{H}}(\hat{P}) = 1$$

For generic n we have

$$P_n = |n\rangle\langle n| \longleftrightarrow \psi_n = f_{n,n}(r,\phi) = (-1)^n 2 L_n\left(\frac{r^2}{\theta}\right) e^{-\frac{r^2}{\theta}}$$

after rescaling back

$$x, y \longrightarrow \frac{x}{\sqrt{\theta}}, \frac{y}{\sqrt{\theta}} \qquad r = \sqrt{x^2+y^2}$$

$$L_m(x) = \sum_{k=1}^{m} \binom{m}{k} \frac{1}{k!} (-x)^k \qquad \text{Laguerre polyn.}$$

One can switch on a background B field.

Ex.: along 24-th, 25-th directions $B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & \end{pmatrix}$

$\hookrightarrow$ $G_{\alpha\beta} = \sqrt{\mathrm{Det}\, G}\, \delta_{\alpha\beta}$ $\qquad \mathrm{Det}\, G = \left(1 + (2\pi B)^2\right)^2$

$$\theta^{\alpha\beta} = -(2\pi\alpha')^2 B\, \epsilon^{\alpha\beta}$$

The canonical commutators change:

$$\left[a_M^{(r)\alpha}, a_N^{(s)\beta\dagger} \right] = G^{\alpha\beta} \delta_{MN} \delta^{rs}$$

The vertex change

$$V_{00} \longrightarrow \mathcal{V}_{00}^{\alpha\beta, rs} = G^{\alpha\beta} \delta^{rs} - \frac{2A^{-1}b}{2a^2+3} \left(G^{\alpha\beta} \phi^{rs} - ia\epsilon^{\alpha\beta} \chi^{rs} \right)$$

$$V_{0n} \longrightarrow \mathcal{V}_{0n}^{\alpha\beta, rs} = \frac{2A^{-1}\sqrt{b}}{4a^2+3} \sum_{t=1}^{3} \left(G^{\alpha\beta} \phi^{rt} - ia\epsilon^{\alpha\beta} \chi^{rt} \right) V_{0n}^{ts}$$

$$V_{mn} \longrightarrow \mathcal{V}_{mn}^{\alpha\beta, rs} = G^{\alpha\beta} V_{mn}^{rs} - \frac{2A^{-1}}{4a^2+3} \sum_{t,v=1}^{3} V_{m0}^{rv} \left(G^{\alpha\beta} \phi^{vt} - ia\epsilon^{\alpha\beta} \chi^{vt} \right) V_{0n}^{ts}$$

where

$$\phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \qquad \chi = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\chi^2 = -2\phi$$
$$\phi\chi = \chi\phi = \frac{3}{2}\chi$$
$$\phi^2 = \frac{3}{2}\phi$$

$$A = V_{00} + \frac{b}{2} \qquad a = -\frac{\pi^2}{A} B$$

Introduce $C_{MN} = (-1)^N \delta_{MN}$

and define $\mathcal{X}^{rs} \equiv C\, V^{rs}$ $\qquad \mathcal{X}^{11} = \mathcal{X}$

then $[\mathcal{X}^{rs}, \mathcal{X}^{r's'}] = 0$

and

$$|\Xi\rangle = \left(\mathrm{Det}(1-X)\,\mathrm{Det}(1+T)\right)^{12} e^{-\frac{1}{2}\eta_{\bar\mu\bar\nu}\sum_{m,n\geq 1} a_m^{\bar\mu +} S_{mn} a_n^{\bar\nu +}} |0\rangle \,\cdot$$

$$\cdot\, \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}\,(\mathrm{Det}\,G)^{1/4}} \sqrt{\mathrm{Det}(1-\mathcal{X})\,\mathrm{Det}(1+\mathcal{T})}\; e^{-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha +} \mathcal{S}_{MN}^{\alpha\beta} a_N^{\beta +}} |\tilde{0}\rangle$$

$\bar\mu, \bar\nu = 0, \ldots, 24$

$\mathcal{S} = C\mathcal{T}$ $\qquad \mathcal{T} = \frac{1}{2\mathcal{X}}\left(1+\mathcal{X} - \sqrt{(1+3\mathcal{X})(1-\mathcal{X})}\right)$

$\mathcal{T}$ is solution of

$$\mathcal{X}\mathcal{T}^2 - (1+\mathcal{X})\mathcal{T} + \mathcal{X} = 0$$

Then

$$|\Xi\rangle * |\Xi\rangle = |\Xi\rangle$$

and

$$\frac{e_{23}}{e_{25}} = \frac{(2\pi)^2}{\sqrt{1+(2\pi B)^2}}\, \mathcal{R}$$

right ratio for D-brane tensions!

$$\mathcal{R} = \frac{A^4(3+4a^2)^2}{2\pi b^3\,(\mathrm{Det}\,G)^{1/4}} \frac{\mathrm{Det}(1-\mathcal{X})^{3/4}\,\mathrm{Det}(1+3\mathcal{X})^{1/4}}{\mathrm{Det}(1-X)^{3/2}\,\mathrm{Det}(1+3X)^{1/2}} = 1$$

Field theory limit: $\alpha' \to 0$

In this limit:

$$V_{00}^{\alpha\beta, rs} \longrightarrow G^{\alpha\beta}\delta^{rs} - \frac{4}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - i a \epsilon^{\alpha\beta}\chi^{rs}\right)$$

$$V_{0n}^{\alpha\beta, rs} \longrightarrow 0$$

$$V_{mn}^{\alpha\beta, rs} \longrightarrow G^{\alpha\beta} V_{mn}^{rs}$$

Introducing

$$|x\rangle = \sqrt{\frac{2\sqrt{\det G}}{b\pi}}\; e^{-\frac{1}{b} x^\alpha G_{\alpha\beta} x^\beta - \frac{2}{\sqrt{b}} i a_0^{\alpha +} G_{\alpha\beta} x^\beta + \frac{1}{2} a_0^{\alpha +} G_{\alpha\beta} a_0^{\beta +}} |\Omega_{b,\theta}\rangle$$

One finds

$$\langle x | S \rangle = \frac{1}{\pi} e^{-\frac{1}{2|a|b} x^\alpha G_{\alpha\beta} x^\beta} |\Xi\rangle$$

$$= \frac{1}{\pi} e^{-\frac{x^\alpha \delta_{\alpha\beta} x^\beta}{\theta}} |\Xi\rangle \qquad \theta = \frac{1}{B}$$

More solutions ?

Then

$$\boxed{\begin{aligned} |\Lambda_n\rangle * |\Lambda_m\rangle &= \delta_{n,m}|\Lambda_n\rangle \\ \langle\Lambda_n|\Lambda_m\rangle &= \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle \end{aligned}}$$

Field theory limit

$$\langle x|\Lambda_n\rangle \longrightarrow \frac{1}{\pi}(-1)^n L_n\left(\frac{x^2+y^2}{\theta}\right) e^{-\frac{x^2+y^2}{\theta}} |\Xi\rangle$$

GMS solitons

Remarkable! Isomorphism between SFT $*$ product and Moyal product

$P_n = |n\rangle\langle n|$ $\qquad\qquad$ $\psi_n(x,y) = \frac{1}{\pi}(-1)^n L_n\left(\frac{x^2+y^2}{\theta}\right) e^{-\frac{x^2+y^2}{\theta}}$

$$|\Lambda_n\rangle \longleftrightarrow P_n \longleftrightarrow \psi_n$$

$$|\Lambda_n\rangle * |\Lambda_{n'}\rangle \longleftrightarrow P_n P_{n'} \longleftrightarrow \psi_n * \psi_{n'}$$

$$\langle\Lambda_n|\Lambda_{n'}\rangle \longleftrightarrow \mathrm{Tr}(P_n P_{n'}) \longleftrightarrow \int dx\,dy\, \psi_n \psi_{n'}$$

Define projectors:

$$\rho_1 = \frac{X^{12}(1-\tau X) + \tau (X^{21})^2}{(1+\tau)(1-X)} \qquad \rho_2 = \frac{X^{21}(1-\tau X) + \tau (X^{12})^2}{(1+\tau)(1-X)}$$

$$\rho_1^2 = \rho_1 \qquad \rho_2^2 = \rho_2 \qquad \rho_1 + \rho_2 = 1$$

Now take two "vectors" ξ and η $\qquad \xi = \{\xi_{N\alpha}\}$

such that $\qquad \eta = \{\eta_{N\alpha}\}$

$$\rho_1 \xi = 0, \quad \rho_2 \xi = \xi \qquad \rho_1 \eta = 0, \quad \rho_2 \eta = \eta$$

Define

$$x = a^+ \tau \xi \quad a^+ C \eta \qquad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

Laguerre pol.

$$|\Lambda_n\rangle = (-\alpha)^n L_n\left(\frac{x}{\alpha}\right)|0\rangle \qquad n = 0, 1, 2, \ldots$$

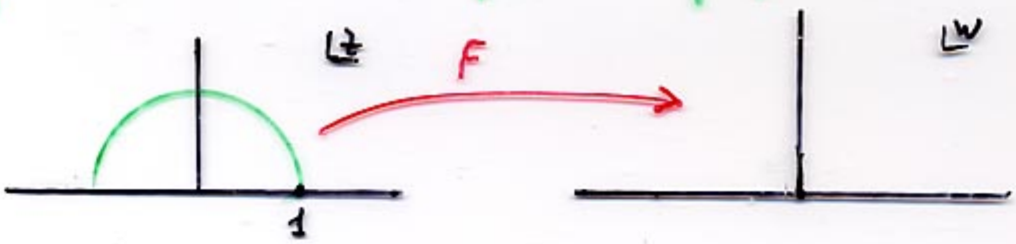
Moreover require

$$\xi \frac{1}{1-\tau^2} \eta = -1, \qquad \xi \frac{\tau}{1-\tau^2} \eta = -\alpha$$

α = const.

- Surface states

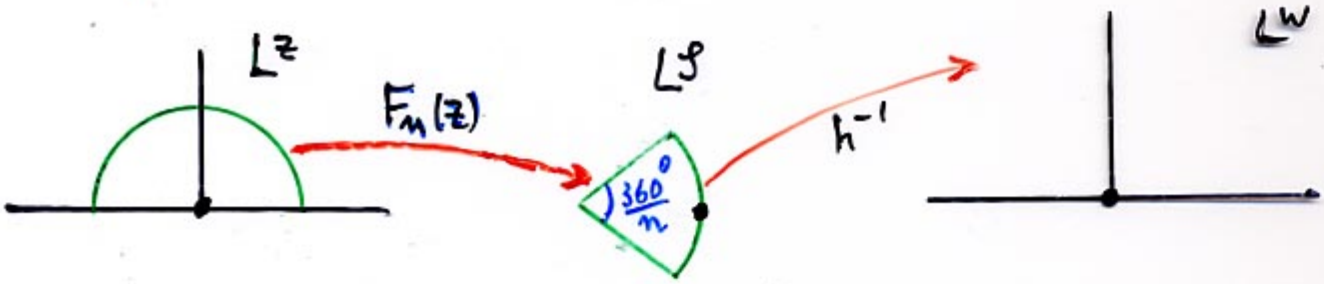
defined via conformal map $F(z)$ of the upper half disk to the upper half plane



$\langle f|$ is defined via

$$\langle f|\phi\rangle = \langle f\circ\phi(0)\rangle \qquad |\phi\rangle = \phi(0)|0\rangle$$

- Wedge states



$$F_n(z) = \left(\frac{1+iz}{1-iz}\right)^{2/n} \qquad h^{-1}(\xi) = -i\,\frac{\xi-1}{\xi+1}$$

$$f_n = h^{-1}\circ F_n(z) = \mathrm{tg}\left(\frac{2}{n}\,\mathrm{arctg}(z)\right)$$

Then

$$|n\rangle * |m\rangle = |n+m-1\rangle$$

and

$$|n=1\rangle = |I\rangle \qquad |n=\infty\rangle = \text{sliver}$$

Representation of wedge states $|n\rangle$

1) $\langle n|\phi\rangle \equiv \langle F_n \circ \phi(0)\rangle$ for any state $|\phi\rangle = \phi(0)|0\rangle$

$$F_n(z) = \frac{n}{2}\,\mathrm{tg}\left(\frac{2}{n}\,\mathrm{tg}^{-1}(z)\right)$$

2) $$|n\rangle = \exp\left(-\frac{n^2-4}{3n^2}L_{-2} + \frac{n^4-16}{30n^4}L_{-4} - \frac{(n^2-4)(176+128n^2+11n^4)}{1890n^6}L_{-6} + \dots\right)|0\rangle$$

Star product of wedge states

$$|n\rangle * |m\rangle = |n+m-1\rangle$$

Two states satisfy $\psi * \psi = \psi$

$n=1$ <u>identity state</u> $|I\rangle \equiv |1\rangle$

$n=\infty$ <u>sliver state</u> $|\Xi\rangle \equiv |\infty\rangle$

$|I\rangle$

$|\infty\rangle$

Using the representation :

$$|\Xi\rangle = e^{\left(-\frac{1}{3}L_{-2}+\frac{1}{30}L_{-4}-\frac{11}{1890}L_{-6}+\frac{34}{467775}L_{-8}+\dots\right)}|0\rangle$$

and $\qquad L_{-n} = L^{m}_{-n} + L^{g}_{-n}$

one gets

$$|\Xi\rangle = |\Xi_g\rangle \otimes |\Xi_m\rangle$$

$$|\Xi_m\rangle = \bar{\mathcal{N}}^{26}\exp\left(-\frac{1}{3}L^{m}_{-2}+\frac{1}{30}L^{m}_{-4}-\frac{11}{1890}L^{m}_{-6}+\dots\right)|0\rangle$$

Then

$$|\Xi_m\rangle *^m |\Xi_m\rangle = K\bar{\mathcal{N}}^{52}|\Xi_m\rangle$$

Now, choose $\bar{\mathcal{N}}$ so that

$$K\bar{\mathcal{N}}^{52} = 1$$

and compare

$$|\Psi_m\rangle = \mathcal{N}^{26}\, e^{-\frac{1}{2}\eta_{\mu\nu}a^{\mu+}\cdot S\cdot a^{\nu+}}|0\rangle$$

with

$$|\Xi_m\rangle = \bar{\mathcal{N}}^{26}\, e^{-\frac{1}{2}\eta_{\mu\nu}a^{\mu+}\cdot \bar{S}\cdot a^{\nu+}}|0\rangle$$

Numerically $\qquad S_{mn} \simeq \bar{S}_{mn}$

The butterfly state

$$\langle B_\alpha | \phi \rangle = \langle f_\alpha \circ \phi(0) \rangle_D$$

where

$$f_\alpha(\xi) = \frac{1}{\alpha} \sin(\alpha \ \mathrm{arctg}\, \xi)$$

When $\alpha \to 0$ we recover the sliver

When $\alpha = 1$ we get the butterfly

$$f_1(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}$$

One can prove

$$|B_\alpha\rangle * |B_\alpha\rangle = |B_\alpha\rangle$$

Split String Field Theory

Treat separately the L and R half of the string.
Define

$$\ell(\sigma) = x(\sigma) \qquad r(\sigma) = x(\pi - \sigma) \qquad 0 \le \sigma \le \frac{\pi}{2}$$

Neumann b.c. at $\sigma = \frac{\pi}{2}$ Dirichlet b.c. at $\sigma = 0, \pi$

Then

$$\begin{cases} \ell(\sigma) = \sqrt{2} \sum_{n=0}^{\infty} \ell_{2n+1} \cos(2n+1)\sigma \\ r(\sigma) = \sqrt{2} \sum_{n=0}^{\infty} r_{2n+1} \cos(2n+1)\sigma \end{cases}$$

and

$$\begin{cases} x_{2n+1} = \frac{1}{2}\left(\ell_{2n+1} - r_{2n+1}\right) \\ x_{2n} = \frac{1}{2} \sum_{k=0}^{\infty} X_{2n,2k+1} \left(\ell_{2k+1} + r_{2k+1}\right) \end{cases}$$

$$\begin{cases} \ell_{2k+1} = x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}\, x_{2n} \\ r_{2k+1} = - x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}\, x_{2n} \end{cases}$$

In this we can define for any $\Psi[x(\sigma)]$ an operator $\hat{\Psi}$

$$\Psi[x(\sigma)] \Longrightarrow \hat{\Psi} = \int \mathcal{D}\ell\, \mathcal{D}r\, |\ell\rangle \Psi[\ell, r] \langle r|$$

also

$$\langle x(\sigma)|\Psi\rangle = \langle \ell|\hat{\Psi}|r\rangle \qquad |\ell\rangle = |\{\ell_{2n+1}\}\rangle$$

In particular

$$\int \Psi \Longrightarrow \mathrm{Tr}(\hat{\Psi}) \qquad \Psi_1 * \Psi_2 \Longrightarrow \hat{\Psi}_1 \hat{\Psi}_2$$

In the half-string formalism the sliver factorizes

$$\langle \vec{x} | = K_0^{26} \langle 0 | \, e^{-x\cdot E^{-2}\cdot x + 2i\,a\cdot E^{-1}\cdot x + \frac{1}{2} a\cdot a}$$

with

$$\hat{x}_n^\mu = \frac{i}{\sqrt{2n}} \left(a_n^\mu - a_n^{\mu+} \right), \qquad \hat{x} = \frac{i}{2} E\cdot(a - a^+) \qquad E_{nm} = \sqrt{\frac{2}{n}}\,\delta_{nm}$$

Then

$$\langle \vec{x} | \Xi \rangle = \tilde{\mathcal{N}}^{26} \, e^{-\frac{1}{2} x\cdot V\cdot x}$$

where

$$V = 2\, E^{-1} \frac{1-S}{1+S} E^{-1}$$

After passing to the half-string basis $x \to (x_L, x_R)$

$$\langle \vec{x} | \Xi \rangle = \tilde{\mathcal{N}}^{26} \, e^{-\frac{1}{2} x^L\cdot K\cdot x^L} \, e^{-\frac{1}{2} x^R\cdot K\cdot x^R}$$

with

$$K = A_+^T V A_+ = A_-^T V A_-$$

and

$$x_n^\mu = A^+_{nm} x_m^{L\mu} + A^-_{nm} x_m^{R\mu} \qquad m, n \geq 1$$

STAR ALGEBRA SPECTROSCOPY

PROBLEM: Diagonalize X, X^{12}, X^{21}, T

Use $K_1 = L_1 + L_{-1} \longrightarrow \quad \mathcal{K}_1 = -(1+z^2)\frac{d}{dz}$

with properties

$$[K_1, X] = [K_1, X^{12}] = [K_1, X^{21}] = [K_1, T] = 0$$

Result:

$$K_1 v^{(\kappa)} = \kappa\, v^{(\kappa)}$$

$-\infty < \kappa < +\infty$

$v^{(\kappa)} = (v_1^{(\kappa)}, v_2^{(\kappa)}, \ldots)$

with

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} v_n^{(\kappa)} z^n = \frac{1}{\kappa}\left(1 - e^{-\kappa \operatorname{tg}^{-1} z}\right)$$

Then

$$X v^{(\kappa)} = \mu(\kappa) v^{(\kappa)}, \qquad \mu(\kappa) = -\frac{1}{1 + 2\cosh\frac{\pi\kappa}{2}}$$

$$X^{12} v^{(\kappa)} = \mu^{12}(\kappa) v^{(\kappa)}, \qquad \mu^{12}(\kappa) = -\left(1 + e^{\frac{\pi\kappa}{2}}\right)\mu(\kappa)$$

$$X^{21} v^{(\kappa)} = \mu^{21}(\kappa) v^{(\kappa)}, \qquad \mu^{21}(\kappa) = -\left(1 + e^{-\frac{\pi\kappa}{2}}\right)\mu(\kappa)$$

$$T v^{(\kappa)} = \tau(\kappa) v^{(\kappa)}, \qquad \tau(\kappa) = -e^{-\frac{\pi|\kappa|}{2}}$$

Remark: $-\frac{1}{3} \leq \mu(\kappa) < 0$, spectrum doubly degenerate except for $\mu(0) = -\frac{1}{3}$

MOYAL REPRESENTATION OF SFT

AIM : Writing VSFT in terms of Moyal $*$ product

First, define

$$o_\kappa^\dagger = -\sqrt{2}\, i \sum_{n=1}^{\infty} v_{2n-1}(\kappa)\, a_{2n-1}^\dagger$$

$$e_\kappa^\dagger = \sqrt{2} \sum_{n=1}^{\infty} v_{2n}(\kappa)\, a_{2n}^\dagger$$

with inverses

$$a_{2n-1}^\dagger = \sqrt{2}\, i \int_0^\infty d\kappa\; v_{2n-1}(\kappa)\; o_\kappa^\dagger$$

$$a_{2n}^\dagger = \sqrt{2} \int_0^\infty d\kappa\; v_{2n}(\kappa)\; e_\kappa^\dagger$$

and commutators

$$[o_\kappa, o_{\kappa'}^\dagger] = [e_\kappa, e_{\kappa'}^\dagger] = \delta(\kappa-\kappa'), \qquad [o_\kappa, e_{\kappa'}^\dagger] = [e_\kappa, o_{\kappa'}^\dagger] = 0$$

The 3-strings vertex becomes

$$|V_3\rangle = \exp\Big[\int_0^\infty d\kappa \Big\{ -\frac{1}{2}\mu(\kappa)\left(o_\kappa^{(1)\dagger} o_\kappa^{(1)\dagger} + e_\kappa^{(1)\dagger} e_\kappa^{(1)\dagger} + \text{cyc.}\right)$$
$$-\frac{1}{2}\left(\mu^{12}(\kappa)+\mu^{21}(\kappa)\right)\left(o_\kappa^{(1)\dagger} o_\kappa^{(2)\dagger} + e_\kappa^{(1)\dagger} e_\kappa^{(2)\dagger} + \text{cyc.}\right)$$
$$-\frac{i}{2}\left(\mu^{12}(\kappa)-\mu^{21}(\kappa)\right)\left(e_\kappa^{(1)\dagger} o_\kappa^{(2)\dagger} - o_\kappa^{(1)\dagger} e_\kappa^{(2)\dagger} + \text{cyc.}\right)\Big\}\Big]|0\rangle$$

Now define combinations

$$\hat{x}_k = \frac{i}{\sqrt{2}}(e_k - e_k^+) = \sqrt{2}\sum_{n=1}^{\infty} v_{2n}(k)\sqrt{2n}\,\hat{x}_{2n}$$

$$\hat{y}_k = \frac{i}{\sqrt{2}}(o_k - o_k^+) = -\sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n-1}(k)}{\sqrt{2n-1}}\,\hat{p}_{2n-1}$$

there are also

$$\hat{z}_k = \frac{1}{\sqrt{2}}(e_k + e_k^+)$$

$$\hat{w}_k = \frac{1}{\sqrt{2}}(o_k + o_k^+)$$

The eigenvalues x_k, y_k

$$\hat{x}_k|x_k\rangle = x_k|x_k\rangle \quad , \quad \hat{y}_k|y_k\rangle = y_k|y_k\rangle$$

are the Moyal conjugate coordinates

$$\boxed{[x_k, y_{k'}]_* = i\,\theta_k\,\delta(k-k')} \qquad \theta_k = 2\,\mathrm{th}\frac{\pi k}{4}$$

Moyal product for string fields:

$$|\Psi\rangle \longrightarrow \underset{\langle x(\sigma)|\Psi\rangle}{\underset{\parallel}{\Psi(\{x_{2n}\},\{x_{2n+1}\})}} \longrightarrow \tilde{\Psi}(\{x_{2n}\},\{p_{2n-1}\}) \longrightarrow \Psi^M(x_k, y_k)$$

Then

$$|\Psi\rangle_1 * |\Psi\rangle_2 \longleftrightarrow \Psi_1^M * \Psi_2^M$$

Witten $\qquad\qquad\qquad\qquad$ Moyal

Sliver takes form

$$|\Xi\rangle = \mathcal{N}^{26}\, e^{-\frac{1}{2}\int_0^\infty d\kappa\, \frac{\theta_\kappa - 2}{\theta_\kappa + 2}\left(e_\kappa^+ e_\kappa^+ + o_\kappa^+ o_\kappa^+\right)} |0\rangle$$

SFT action and properties of Q_B

$$S(\Phi) = -\frac{1}{g_0^2}\left[\frac{1}{2}\langle \Phi, Q_B \Phi\rangle + \frac{1}{3}\langle \Phi, \Phi * \Phi\rangle\right.$$

BRST charge Q_B:

$$Q_B^2 = 0$$

$$Q_B(A*B) = (Q_B A)*B + (-1)^{|A|} A*(Q_B B)$$

$$\langle Q_B A, B\rangle = -(-1)^{|A|}\langle A, Q_B B\rangle$$

Inner product:

$$\langle A, B\rangle = (-1)^{|A||B|}\langle B, A\rangle$$

$$\langle A, B*C\rangle = \langle A*B, C\rangle$$

Associative $*$ product

$$A*(B*C) = (A*B)*C$$

$|A|$ is the Grassmannality of A

The ghost sector (Hata, Kawano)

After factorization we have to solve

$$\mathcal{Q}\,\Psi_g + \Psi_g *^g \Psi_g = 0$$

where

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n \mathcal{C}_n \qquad\qquad \mathcal{C}_n = c_n + (-1)^n c_{-n}$$

The vacua are

$$|\dot{0}\rangle = c_1|0\rangle \qquad\qquad |\hat{0}\rangle = c_0 c_1|0\rangle$$

and the 3-strings vertex

$$|V_3\rangle = e^{\sum_{n,m\geq 1} c_n^{(r)\dagger}\tilde{V}_{nm}^{rs} b_m^{(s)\dagger} + \sum_{n\geq 1} c_n^{(r)\dagger}\tilde{V}_{n0}^{rs} b_0^{(s)\dagger}}|\hat{0}\rangle_1|\hat{0}\rangle_2|\hat{0}\rangle_3$$

The ansatz for Ψ_g is

$$|\Psi_g\rangle = b_0|\dot{\phi}_g\rangle \qquad\qquad (\text{Siegel gauge: } b_0|\Psi_g\rangle = 0)$$

$$|\dot{\phi}_g\rangle = \mathcal{N}_g\, e^{\sum_{n,m\geq 1} c_n^\dagger \tilde{S}_{nm} b_m^\dagger}|\dot{0}\rangle$$

One finds that

$$\tilde{T} = C\tilde{S} \qquad\qquad \tilde{T} = \frac{1}{2\tilde{X}}\left(1+\tilde{X} - \sqrt{(1+3\tilde{X})(1-\tilde{X})}\right)$$

$$\vec{f} = \frac{1}{1-\tilde{T}}\left[\vec{y} + (\tilde{X}_+, \tilde{X}_-)\frac{1}{1-\tilde{T}\tilde{\mathcal{X}}}\tilde{T}\begin{pmatrix}\vec{y}_+\\ \vec{y}_-\end{pmatrix}\right]$$

where

$$\vec{f} = \{f_n\}$$

$$\widetilde{\mathcal{M}} = \begin{pmatrix} \widetilde{X} & \widetilde{X}_+ \\ \widetilde{X}_- & \widetilde{X} \end{pmatrix} \qquad \widetilde{\mathcal{T}} = \begin{pmatrix} \widetilde{T} & 0 \\ 0 & \widetilde{T} \end{pmatrix}$$

It is not hard to prove that

- $f_{2n+1} = 0 \qquad f_{2n} = 1$

This means

$$\mathcal{Q} = \frac{1}{2}\left(C(i) + C(-i)\right)$$

Midpoint insertion (in twisted theory).

Ghost Neumann coefficients

$$\tilde{V}^{rs}_{nm} = -(-1)^{n+m} \cdot \tilde{N}^{rs}_{nm}$$

where

$$|\hat{0}\rangle = c_0 c_1 |0\rangle$$

$$|\dot{0}\rangle = c_1 |0\rangle$$

$$\tilde{N}^{rs}_{nm} = \langle \tilde{V}_{123} | b^{(r)}_{-n} c^{(s)}_{-m} |\dot{0}\rangle_{123}$$

$$= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n-1}} \frac{1}{w^{m+2}} \left(f_r'(z)\right)^2 \frac{-1}{f_r(z) - f_s(w)} \cdot$$

$$\cdot \prod_{i=1}^{3} \frac{f_s(w) - \xi_i}{f_r(z) - \xi_i} \left(f_s'(w)\right)^{-1}$$

SL(2,R)-invariant bc propagator

$$\langle b(z)\, c(w) \rangle = \frac{1}{z-w} \prod_{i=1}^{3} \frac{w-\xi_i}{z-\xi_i}$$

We choose

$$\xi_i = f_i(0) = \alpha^{2-i}, \qquad \alpha^3 = 1$$

then

$$\prod_{i=1}^{3} \frac{f_s(w) - f_i(0)}{f_r(z) - f_i(0)} = \frac{f^3(w) - 1}{f^3(z) - 1} \qquad \forall\, r, s = 1, 2, 3$$

Decomposition

$$\tilde{N}^{rs}_{nm} = \frac{1}{3}\left(\tilde{E}_{nm} + \bar{\alpha}^{r-s}\, \tilde{U}_{nm} + \alpha^{r-s}\, \tilde{\bar{U}}_{nm}\right)$$

$$\tilde{E}_{nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left(\frac{1}{1+zw} - \frac{w}{w-z}\right)$$

$$\tilde{U}_{nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left(\frac{1}{1+zw} - \frac{w}{w-z}\right) \frac{f(z)}{f(w)} = (-1)^{n+m} \tilde{\bar{U}}_{nm}$$

Properties

$$\tilde{N}^{rs}_{nm} = \tilde{N}^{r+1,s+1}_{nm}$$

$$\tilde{N}^{rs}_{mn} = (-1)^{n+m}\,\tilde{N}^{sr}_{nm}$$

Ambiguity

$\tilde{N}^{rs}_{nm}$ with $\overset{r=s}{-1 \le n, m \le 1}$ are ambiguous

Fix ambiguity:

$$\tilde{N}^{rr}_{-1,1} = \tilde{N}^{rr}_{1,-1} = 0 \qquad N^{rr}_{0,0} = 1$$

Then

$$\boxed{\sum_{k=0}^{\infty} \tilde{U}_{nk}\,\tilde{U}_{km} = \delta_{nm}}$$

Consequences: define

$$\tilde{X}^{rs} = C\,\tilde{V}^{rs}$$

Then

$$\left[\tilde{X}^{rs}, \tilde{X}^{r's'}\right] = 0$$

Call $\mathcal{Y}$ any $\tilde{X}^{rs}$

$$\mathcal{Y} = \left|\begin{array}{c:c} 1 & 0 \\ \hdashline \vec{y} & Y \end{array}\right| \qquad Y = \{ Y_{nm} \,,\; n, m \ge 1 \}$$

Set

$$y \equiv \tilde{X}^{11}$$
$$y_+ \equiv \tilde{X}^{12}$$
$$y_- \equiv \tilde{X}^{21}$$

Then

$$y + y_+ + y_- = 1$$
$$y^2 + y_+^2 + y_-^2 = 1$$
$$y_+^3 + y_-^3 = 2y^3 - 2y^2 + 1$$
$$y_- y_+ = y^2 - y$$
$$[y, y_\pm] = 0$$
$$[y_+, y_-] = 0$$

which decompose into

$$y + y_+ + y_- = 1 \qquad \vec{y} + \vec{y}_+ + \vec{y}_- = 0$$
$$y^2 + y_+^2 + y_-^2 = 1 \qquad (1+y)\vec{y} + y_+\vec{y}_+ + y_-\vec{y}_- = 0$$
$$y_+^3 + y_-^3 = 2y^3 - 3y^2 + 1 \qquad y_+^2\vec{y}_+ + y_-^2\vec{y}_- = (2y^2 - y - 1)\vec{y}$$
$$y_+ y_- = y^2 - y \qquad y_+\vec{y}_- = y\vec{y} = y_-\vec{y}_+$$
$$[y, y_\pm] = 0 \qquad -y_\pm\vec{y} = (1-y)\vec{y}_\pm$$
$$[y_+, y_-] = 0$$

Fluctuation spectrum (Hata, Kawano)

Fluctuations are the solutions of linearized equations

$$\phi = \phi_0 + \tilde{\phi} \qquad\qquad 2\tilde{\phi} + \phi_0 * \tilde{\phi} + \tilde{\phi} * \phi_0 = 0$$

Let us look for a tachyon solution $|\tilde{\phi}\rangle = b_0 |\phi_t\rangle$

with $p^2 = 1 \quad (\alpha' = 1)$

$$|\phi_t\rangle = \mathcal{N}_t \exp\Big[-\frac{1}{2}\sum_{n,m\geq 1} a_n^\dagger S_{nm} a_m^\dagger + \sum_{n,m\geq 1} c_n^\dagger \tilde{S}_{nm} b_m^\dagger - \sum_{n\geq 1}\sqrt{2}\, t_n a_n^\dagger \cdot p\Big] |0\rangle$$

and

$$Ct = t$$

Also

$$|\phi_t\rangle = \frac{\mathcal{N}_t}{\mathcal{N}_0} e^{-\sqrt{2}\sum_{n\geq 1} t_n a^\dagger \cdot p} |\phi_0\rangle \qquad\qquad |\phi_0\rangle \equiv \text{sliver}$$

Explicit computation gives

$$\left.{}_1\langle\phi_0|\,{}_2\langle\phi_t|\,\hat{V}\rangle_{123}\right|_{p_1=0,\, p_2=-p_3} = -\frac{\mathcal{N}_t}{\mathcal{N}_0} 2\, e^{-\sqrt{2}\sum_{n\geq 1} u_n a_n^{(3)\dagger}\cdot p^{(3)} - p^2 G} |\phi_0\rangle_3$$

$$= -|\phi_t\rangle_3$$

This implies

$$\vec{t} = \vec{u} = -\vec{v}_0 + \vec{v}_+ - (V_+, V_-)\frac{1}{1-\mathcal{S}\mathcal{V}}\mathcal{S}\begin{pmatrix}\vec{v}_+ - \vec{v}_-\\ \vec{v}_- - \vec{v}_0\end{pmatrix} + (V_+, V_-)\frac{1}{1-\mathcal{S}\mathcal{V}}\begin{pmatrix}0\\ \vec{t}\end{pmatrix}$$

$$V_\pm = \begin{matrix}V^{12}\\ V^{21}\end{matrix} \qquad \vec{v}_0 = \{V^{11}_{0n}\}, \quad \vec{v}_+ = \{V^{12}_{0n}\} \qquad \vec{v}_- = \{V^{21}_{0n}\}$$

and

$$G = 2V_{00} + \left(\vec{v}_+ - \vec{v}_-\,,\ \vec{v}_- - \vec{v}_0\right)\frac{1}{1-\mathcal{S}\mathcal{V}}\,\mathcal{S}\begin{pmatrix}\vec{v}_+ - \vec{v}_-\\ \vec{v}_- - \vec{v}_0\end{pmatrix} -$$

$$- 2\left(\vec{v}_+ - \vec{v}_-,\ \vec{v}_- - \vec{v}_0\right)\frac{1}{1-\mathcal{S}\mathcal{V}}\begin{pmatrix}0\\ \vec{t}\end{pmatrix} + \left(0, \vec{t}\right)\mathcal{V}\frac{1}{1-\mathcal{S}\mathcal{V}}\begin{pmatrix}0\\ \vec{t}\end{pmatrix}$$

$$\mathcal{S} = \begin{pmatrix}S & 0\\ 0 & S\end{pmatrix} \qquad \mathcal{V} = \begin{pmatrix}V_0 & V_+\\ V_- & V_0\end{pmatrix}$$

Simplifying

$$\vec{t} = -3\,\frac{T^2-T+1}{T+1}\,\vec{v}_0$$

$$G = 2V_{00} - 9\,\vec{v}_0\,\frac{T^2-T+1}{(T+1)^2}\,\vec{v}_0 = 0 \quad !!$$

because

$$\frac{2}{9}V_{00} = \vec{v}_0\,\frac{T^2-T+1}{(T+1)^2}\,\vec{v}_0 \qquad\qquad V_{00} = \operatorname{Log}\frac{27}{16}$$

But we expect

$$e^{-Gp^2} = e^{-G} = \frac{1}{2}$$

Due to insertion at midpoint!

Remedy: - smear along the boundary and use BCFT (Okawa)

- regularize by level truncation (Hata et al.)

Vector state

$$|\phi_v\rangle = \sum_{\substack{n \\ \text{odd}}} d_n \cdot a_n^\dagger |\phi_t\rangle$$

with

$$C d = -d$$

Satisfies EOM, but no transversality.

SUMMARY OF VSFT

- D-25 brane solution exists
- It has the right spectrum
- D-(25-k) brane solutions exist
- Background B-Field : OK $\longrightarrow$ GMS solitons
- Split SFT
- Moyal SFT

All results are analytic!

PROBLEMS

- The vacuum solution Φ_0 in Witten's SFT
- The nature of tachyon condensation (closed strings)
- Closed SFT
- Higher loops corrections (W. Taylor et al.)
- SSFT (brane solution?)

Closed String Field Theory

- Covariant CSFT (B. Zwiebach)
- Non-covariant CSFT
- HIKKO CSFT (is covariant but depends on string-length parameter α)

Boundary state in this theory is nilpotent.

$$|B\rangle * |B\rangle = |B\rangle$$

(I. Kishimoto, Y. Matsuo, E. Watanabe, hep-th/0306189)

HIKKO (H. Hata, K. Itoh, T. Kugo, H. Kunitomo, K. Ogawa)

Integrable structures in SFT

Hirota equations for dTL (dispersionless Toda lattice hierarchy):

H1: $$(z_1 - z_2)\, e^{D(z_1) D(z_2) F} = z_1\, e^{-\partial_{t_0} D(z_1) F} - z_2\, e^{-\partial_{t_0} D(z_2) F}$$

H2: $$z_1 \bar{z}_2 \left(1 - e^{-D(z_1) \bar{D}(\bar{z}_2) F}\right) = e^{\partial_{t_0}\left(\partial_{t_0} + D(z_1) + \bar{D}(\bar{z}_2)\right) F}$$

where

$$D(z) = \sum_{k=1}^{\infty} \frac{1}{k z^k} \frac{\partial}{\partial t_k} \qquad \bar{D}(\bar{z}) = \sum_{k=1}^{\infty} \frac{1}{k \bar{z}^k} \frac{\partial}{\partial \bar{t}_k}$$

and

$$F = \ln \tau$$

is the free energy of the system.

A. Boyarsky, O. Ruchayskiy : hep-th/0211010

L. B. , A. Sorin : hep-th/0211283

Use definition of 3-string vertex (LeClair-Peskin-Preitsch.)

$$N^{21}_{nm} = -\frac{1}{n}\oint_0 \frac{dz}{2\pi i}\frac{1}{z^n} f_2'(z)\,\frac{1}{m}\oint \frac{dw}{2\pi i}\frac{1}{w^m} f_1'(w)\,\frac{1}{(f_2(z)-f_1(w))^2}.$$

where

$$f_1(z) = e^{\frac{2\pi i}{3}}\left(\frac{1+iz}{1-iz}\right)^{2/3}$$

$$f_2(z) = \left(\frac{1+iz}{1-iz}\right)^{2/3}$$

$$f_3(z) = e^{-\frac{2\pi i}{3}}\left(\frac{1+iz}{1-iz}\right)^{2/3}$$

and identify

$$F_{t_n t_m} \equiv \frac{\partial^2 F}{\partial t_n \partial t_m} = -\sqrt{nm}\,V^{11}_{nm} = nm\,N^{11}_{nm}$$

$$F_{t_n \bar{t}_m} \equiv \frac{\partial^2 F}{\partial t_n \partial \bar{t}_m} = -\sqrt{nm}\,V^{12}_{nm} = nm\,N^{12}_{nm}$$

$$F_{t_0 \bar{t}_n} \equiv \frac{\partial^2 F}{\partial t_0 \partial \bar{t}_n} = \sqrt{\frac{n}{2}}\left(V^{12}_{0n} - V^{22}_{0n}\right) = n\left(N^{12}_{0n} - N^{12}_{0n}\right)$$

$$F_{t_0 t_n} \equiv \frac{\partial^2 F}{\partial t_0 \partial t_n} = \sqrt{\frac{n}{2}}\left(V^{21}_{0n} - V^{11}_{0n}\right) = n\left(N^{11}_{0n} - N^{21}_{0n}\right)$$

One can prove that these quantities satisfy the Hirota eqs. H1 – H2.

All classical solutions (such as the sliver) satisfy Hirota eq.

H1:

$$F_{t_1t_1} = \frac{1}{2} F_{t_0t_2} - \frac{1}{2} \left(F_{t_0t_1}\right)^2$$

$$\frac{1}{2} F_{t_1t_2} = \frac{1}{3} F_{t_0t_3} - \frac{1}{2} F_{t_0t_1} F_{t_0t_2} + \frac{1}{6} \left(F_{t_0t_1}\right)^3$$

$$\frac{1}{4} F_{t_2t_2} + \frac{1}{2} \left(F_{t_1t_1}\right)^2 - \frac{1}{3} F_{t_1t_3} = 0$$

$$\frac{1}{3} F_{t_1t_3} = \frac{1}{4} F_{t_0t_4} - \frac{1}{3} F_{t_0t_1} F_{t_0t_3} - \frac{1}{8} \left(F_{t_0t_2}\right)^2 + \frac{1}{4} F_{t_0t_1}^2 F_{t_0t_2} - \frac{1}{24} F_{t_0t_1}^4$$

$$\vdots$$

H2:

$$F_{t_1\bar{t}_1} = e^{F_{t_0t_0}}$$

$$F_{t_2\bar{t}_1} = 2\, e^{F_{t_0t_0}} F_{t_1t_0}$$

$$F_{t_3\bar{t}_1} = \frac{3}{2} e^{F_{t_0t_0}} F_{t_2t_0} + \frac{3}{2} e^{F_{t_0t_0}} F_{t_1t_0}^2$$

$$\frac{1}{4} F_{t_2\bar{t}_2} - \frac{1}{2} F_{t_1\bar{t}_1}^2 = e^{F_{t_0t_0}} F_{t_1t_0} F_{t_0\bar{t}_1}$$

$$F_{t_0t_0} = \ln \frac{16}{27},$$

$$F_{t_1\bar{t}_1} = \frac{16}{27}, \qquad F_{t_1\bar{t}_2} = \frac{64}{81\sqrt{3}}, \qquad F_{t_3\bar{t}_1} = -\frac{16}{35}$$

Problem: determine 2-matrix model

Developments:

- Higher genus calculations
- TLH (dispersive) valid for all genera
- 2-matrix model is an all-genus model

Hirota eqs. for dispersive TLH (Zabrodin '01)

$$z_1 \left(e^{\hbar(\partial_{t_0} - D(z_1))} \tau_\hbar \right) \left(e^{-\hbar D(z_2)} \tau_\hbar \right) -$$
$$- z_2 \left(e^{\hbar(\partial_{t_0} - D(z_2))} \tau_\hbar \right) \left(e^{-\hbar D(z_1)} \tau_\hbar \right) =$$
$$= (z_1 - z_2) \left(e^{-\hbar(D(z_1) + D(z_2))} \tau_\hbar \right) \left(e^{\hbar \partial_{t_0}} \tau_\hbar \right)$$

$$\left(e^{-\hbar D(z_1)} \tau_\hbar \right) \left(e^{-\hbar \bar{D}(\bar{z}_2)} \tau_\hbar \right) - \tau_\hbar \left(e^{\hbar(\bar{D}(\bar{z}_2) - D(z_1))} \tau_\hbar \right) =$$
$$= \frac{1}{z_1 \bar{z}_2} \left(e^{-\hbar(\partial_{t_0} + D(z_1))} \tau_\hbar \right) \left(e^{\hbar(\partial_{t_0} + \bar{D}(\bar{z}_2))} \tau_\hbar \right)$$

$$\log \tau_\hbar = F_\hbar$$

Rolling tachyon &

open-closed string duality

A rolling tachyon is a classical solution of SFT which represents the evolution in time of the tachyon field $T(x^0)$. (Sen)

In classical field theory

$$T(x^0) = \lambda \cosh x^0$$

$T(0) = \lambda$
$T'(0) = 0$

In CFT

$$-\frac{1}{2\pi} \int_\Sigma d^2z \, \partial_z X^0 \partial_{\bar z} X^0 + \tilde\lambda \int_{\partial\Sigma} dt \cosh X^0(t)$$

with $\tilde\lambda \simeq \lambda$.

Wick rotate $X_0 \to iX$ and study boundary state $|B\rangle$ perturbed by

$$\tilde\lambda \int dt \cos X(t)$$

After inverse-Wick rotating, the relevant part of $|B\rangle$ is

$$|B\rangle_{c=1} \sim \left[f(X^0(0)) + \alpha_{-1}^0 \tilde{\alpha}_{-1}^0 \, g(X^0(0)) \right] |0\rangle$$

where

$$f(x^0) = \frac{1}{1+e^{x^0}\sin\tilde{\lambda}\pi} + \frac{1}{1+e^{-x^0}\sin\tilde{\lambda}\pi} - 1$$

and

$$g(x^0) = \cos(2\pi\tilde{\lambda}) + 1 - f(x^0)$$

Interpretation

$$T_{00} = K\left(f(x^0) + g(x^0)\right) = K\left(\cos(2\pi\tilde{\lambda}) + 1\right)$$

$$T_{0i} = 0$$

$$T_{ij} = -2K\, f(x^0)\, \delta_{ij} \qquad K = \frac{1}{2}\tau_p$$

Comments.

- For $\tilde{\lambda} = \frac{1}{2}$ total energy vanishes, $f(x^0) = 0$ (equivalent to array of D-branes at $x^0 = i(2n+1)\pi$)
- $0 < \tilde{\lambda} < \frac{1}{2}$ system evolves.

$$f(x^0) \xrightarrow[x^0 \to \infty]{} 0 \qquad g(x^0) \longrightarrow 1 + \cos(2\pi\tilde{\lambda})$$

So

$$T_{00} = \text{const} \qquad T_{ij} \to 0$$

Tachyon matter

Effective Field Theory of Tachyonic Matter (Sen)

Proposal

$$S = -\int d^{p+1}x\, V(T)\, \sqrt{-\det A}\,, \qquad V(T) = e^{-\frac{T}{2}}$$

$$A_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu T\, \partial_\nu T \qquad \mu,\nu = 0,\dots,p$$

For spatially homogeneous time-dependent field configurations

$$T_{00} = e^{-\frac{T}{2}}\left(1-(\partial_0 T)^2\right)^{-1/2}$$

Since T_{00} is conserved, $\partial_0 T \to 1$ as $T \to \infty$.

Solution for large x^0:

$$T = x^0 + C\, e^{-x^0} + O(e^{-2x^0})$$

Pressure:

$$p = e^{-\frac{T}{2}}\sqrt{1-(\partial_0 T)^2} \simeq -\sqrt{2C}\, e^{-x^0}$$

No plane wave solution. Candidate to represent tachyon condensation.

What is "tachyon matter"? (Gaiotto, Itzhaki, Rastelli)

The marginal deformation

$$\tilde{\lambda} \int_{\partial\Sigma} dt \, \cosh X^0(t)$$

seems to represent, for $\tilde{\lambda} = \frac{1}{2}$, the tachyon vacuum.

It is equivalent to an infinite array of D-branes at

$$X^0 = i\,(2n+1)\pi a$$

Relation between Wick-rotated amplitudes $\hat{A}(E)$ and inverse-Wick-rotated amplitudes $S(E)$

$$S(E) = \frac{1}{2\sinh\frac{aE}{2}} \; \mathrm{Disc}_E\left[\hat{A}(iE)\right]$$

$$\mathrm{Disc}_E\, f(E) = \frac{1}{i}\left(f(E+i\epsilon) - f(E-i\epsilon)\right)$$

For instance, 2-point closed string tachyon on the disk $= \{ z,\ |z| \geq \rho \}$

$$\tilde{A}(p_1, p_2) = \int_0^1 d\rho\ \rho^{t/2-3}\left(1-\rho^2\right)^{s-2} = \frac{\Gamma(t/4-1)\,\Gamma(s-1)}{2\,\Gamma\left(\frac{t}{4}+s-2\right)}$$

Open string channel $\quad s = 1, 0, -1, \ldots$

Closed string channel $\quad t = 4, 0, -4, \ldots$

$$S(p_1, p_2) = \frac{1}{2\sinh\frac{a|E|}{2}} \sum_{k=0}^{\infty} f_k(s)\, \delta\left(\frac{t}{4} - 1 + k\right)$$

- S has poles in t, not in s (no open string poles)
- contributions to S come from $\rho \simeq 0$ (disk $\rightarrow$ sphere)

Conclusion $\quad$ S describes a sphere amplitude with two tachyon insertions and a tower of on-shell massive closed string states.

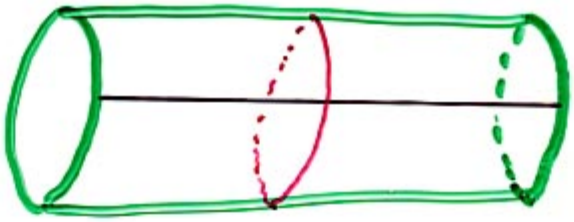
BCFT description

$$|W\rangle \equiv \frac{\delta(L_0 + \bar{L}_0)}{2\sinh\frac{a|E|}{2}}\left(b_0 + \bar{b}_0\right)|B^{p-1}\rangle_{|z|=1}$$

$$|B^{p-1}\rangle = \mathcal{N}\,\delta(x^M)\ e^{+a_n^{\mu\dagger}\eta_{\mu\nu}\tilde{a}_n^{\nu\dagger} - a_n^{m\dagger}\delta_{mp}\tilde{a}_n^{p\dagger}}|0\rangle$$

At $a = 2\pi$ ($\tilde{\lambda} = \frac{1}{2}$) the energy is stored in very massive closed string modes, which behave non-relativistically and are localized near the brane $\Longrightarrow$ tachyon matter

There seems to be an open-closed string duality at tree level.



traditional duality

"Tree level open strings know about closed strings"

REFERENCES

Recent reviews

K. Ohmori "A review on Tachyon Cond.." hepth/0102085

L. Rastelli, A. Sen, B. Zwiebach "Vacuum String Field Theory" hepth/0106010

I. Ya. Aref'eva et al. "Noncommutative FT and (Super) SFT" hepth/0111208

L. B. et al. "Topics in SFT" hepth/0305

Tachyon condensation

A. Sen "Tachyon condensation on the Brane-Antibrane System" hepth/9805175

"Universality of the Tachyon Potential" hepth/9911116

Basic References

E. Witten "Non-commutative Geometry and String Field Theory" NPB 268 (1986) 253

D. Gross, A. Jevicki "Operator Formulation of Interacting String Field Theory" NPB 283 (1987) 1, NPB 287 (1987), 225

A. LeClair, M.E. Peskin and C.R. Preitschopf NPB 317 (1987) 411, NPB 317 (1989) 464

Vacuum String Field Theory

V. A. Kostelecky, R. Potting, "Analytical construction of a nonperturbative vacuum..." hepth/0008252

L. Rastelli, A. Sen, B. Zwiebach:

"String Field Theory around the tachyon vacuum" hepth/0012251

"Classical solutions in SFT..." hepth/0102112

"Half-strings, Projectors, ..." hepth/0105058

"Star algebra spectroscopy" hepth/0111281

D. Gaiotto, L. Rastelli, A. Sen, B. Zwiebach

"Ghost structure ..." hepth/0111129

D. J. Gross, W. Taylor

"Split string field theory" I. II hepth/0105059
0106036

H. Hata, T. Kawano "Open string states ..." hepth/0108150

K. Okuyama "Ghost kinetic operator..." hepth/0201015

"Ratio of tensions..." hepth/0201136

N. Moeller "Some exact results..." hepth/0110204

Y. Okawa "Open string states..." hepth/0204012

Moyal representation of SFT

I. Bars "Map of Witten's * to Moyal's *", hepth/0106157

M. Douglas, H. Liu, G. Moore, B. Zwiebach "Open string star ..." hepth/0202087

B. Feng, Y.-H. He, N. Moeller hepth/0203175

B. Chen, F.L. Lin hepth/0204233

D.M. Belov hepth/0204164

SFT with B field

E. Witten hepth/0006071

M. Schnabl hepth/0010034

F. Sugino hepth/9912254

T. Kawano, T. Takahashi hepth/9912274

L. B., D. Mamone, M. Salizzoni hepth/0201060, 0203188, 0207044

Rolling tachyon

A. Sen "Rolling tachyon" hepth/0203211

A. Sen "Tachyon matter" hepth/0203265

A. Sen "Field theory of tachyon matter" 0204143

0207105
0209122
0208142

A. Sen, P. Mukhopadhyay

A. Sen "Open and Closed Strings from unstable D-branes" hepth 0305011

A. Sen "Open-Closed Duality at Tree Level" hepth/0306137

N. Lambert, H. Liu, J. Maldacena "Closed strings from decaying D-branes" hepth/0303139

D. Gaiotto, N. Itzhaki, L. Rastelli "Closed strings as Imaginary D-branes" hepth/0304192