PATH INTEGRALS ON NONCOMMUTATIVE AND p-ADIC SPACES

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1. Introduction
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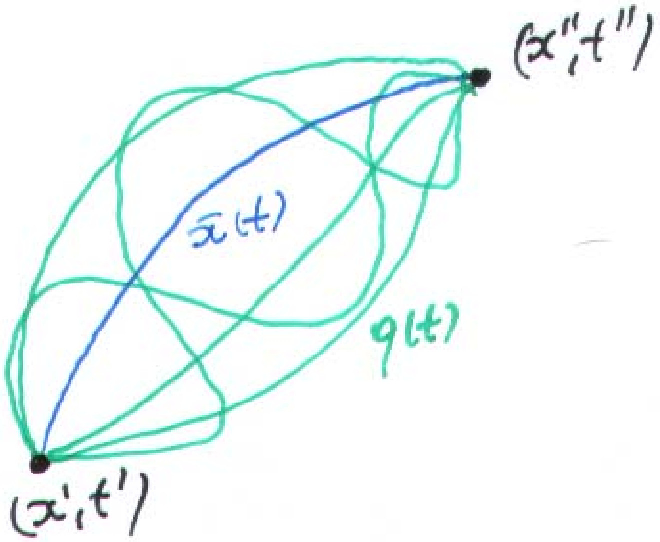
1. Introduction

$\Delta x_i \Delta p_j \geq \frac{\hbar}{2} \delta_{ij}$ standard QM

$\Delta x_i \Delta x_j \geq \frac{\hbar}{2} \theta_{ij}$ noncomm. QM (NCQM)

$\Delta x \geq l_0 = \sqrt{\frac{\hbar G}{c^3}}$ p-adic QM

2. Feynman's path integrals in standard QM



$\Delta x = 0$
$\Delta p = \infty$

$$\mathcal{K}(x'',t'';x',t') = \sum_q \exp \frac{2\pi i}{h} S[q]$$

$$= \int \exp\left(\frac{2\pi i}{h} \int_{t'}^{t''} L(\dot{q},q,t)\,dt\right) \mathcal{D}q$$

probability amplitude

$|\mathcal{K}(x'',t'';x',t')|^2$ probability (density)

- $\mathcal{K}(x'',t'';x',t')$ primary object in Feynman's QM
- Feynman's QM is equivalent to Schrödinger's and Heisenberg's QM
- Dirac (1932), Feynman (1941, ..., 1948)

$$\psi(x'',t'') = \int \mathcal{K}(x'',t'';x',t')\,\psi(x',t')\,dx'$$

(i) $\int_{-\infty}^{+\infty} \mathcal{K}(x'',t'';x,t)\,\mathcal{K}(x,t;x',t')\,dx = \mathcal{K}(x'',t'';x',t')$ (comp. l.)

(ii) $\int_{-\infty}^{+\infty} \mathcal{K}^*(x'',t'';x',t')\,\mathcal{K}(y,t'';x',t')\,dx' = \delta(x''-y)$ (unitarity cond.)

(iii) $\mathcal{K}(x'',t';x',t') = \lim_{t''\to t'} \mathcal{K}(x'',t'';x',t') = \delta(x''-x')$ (initial cond.)

Methods of evaluation of path int.

- time discretization (Feynman)
- analytic continuation, $t \to -i\tau$ (M. Katz)
- evaluation by Fourier series
- Taylor expansion of $S[q]$

$$S[q] = S[\bar{x}+y] = S[\bar{x}] + \underset{0}{\underset{\shortparallel}{\delta S[\bar{x}]}} + \frac{1}{2}\delta^2 S[\bar{x}] + \cdots$$

$$= S[\bar{x}] + \frac{1}{2}\int_{t'}^{t''} \left(\dot{y}\frac{\partial}{\partial\dot{q}} + y\frac{\partial}{\partial q}\right)^2 L(\dot{q},q,t)\,dt + \cdots$$

$y''=y'=0$

$$\mathcal{K}(x'',t'';x',t') = N(t'',t')\exp\frac{2\pi i}{h}\bar{S}(x'',t'';x',t')$$

(ii) + (i) quadratic Lagrangians $L(\dot{q},q,t)$

$$\mathcal{K}(x'',t'';x',t') = \sqrt{\frac{1}{ih}\left(-\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right)}\exp\frac{2\pi i}{h}\bar{S}(x'',t'';x',t')$$

$$\boxed{\mathcal{K}(x'',t'';x',t') = \sqrt{\frac{1}{(ih)^D}\det\left(-\frac{\partial^2\bar{S}}{\partial x''_i\partial x'_j}\right)}\exp\frac{2\pi i}{h}\bar{S}(x'',t'';x',t')}$$

D-dim. path integral for quad. Lagrangians

semiclassical approximation

phase space path int.

$$K(x'',t'';x',t') = \int \exp \frac{2\pi i}{h} \int_{t'}^{t''} [p\dot{q} - H(p,q,t)]\, dt \, \frac{\mathcal{D}q\,\mathcal{D}p}{h}$$

quad. Lagrangians:

phase space PI = config. space PI

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PATH INTEGRAL APPROACH TO NONCOMMUTATIVE QUANTUM MECHANICS

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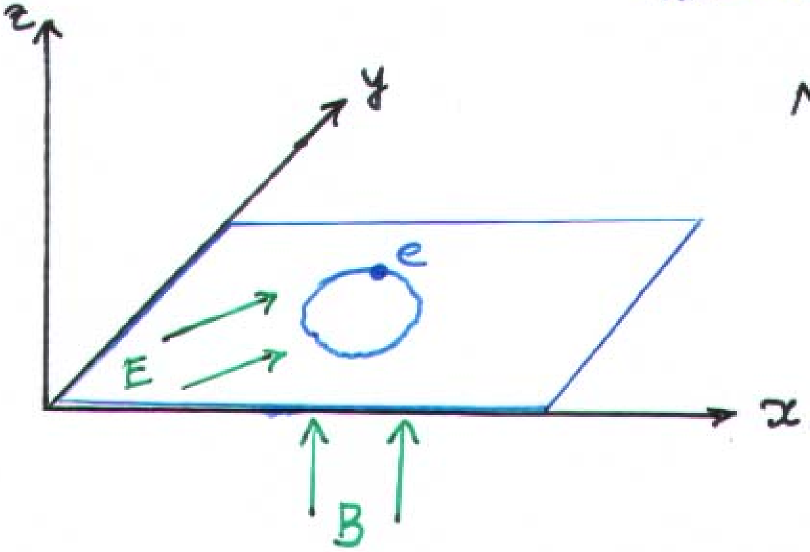
1. Introduction

- significant interest in NC quantum models (string theory, QFT, QM)
- $[\hat{x}_k, \hat{p}_\ell] = i\hbar\delta_{k\ell}$ —
  - $[\hat{x}_k, \hat{x}_\ell] = 0, \quad [\hat{p}_k, \hat{p}_\ell] = 0$
  - $[\hat{x}_k, \hat{x}_\ell] \neq 0, \quad [\hat{p}_k, \hat{p}_\ell] = 0$
  - $[\hat{x}_k, \hat{x}_\ell] \neq 0, \quad [\hat{p}_k, \hat{p}_\ell] \neq 0$

2. Noncommutative QM (NCQM)

$$[\hat{x}_k, \hat{p}_l] = i\hbar\delta_{kl}, \quad [\hat{x}_k, \hat{x}_l] = i\hbar\theta_{kl}, \quad [\hat{p}_k, \hat{p}_l] = 0$$

$$\theta_{kl} = \theta\varepsilon_{kl}$$



NCQM in the plane

$$\begin{cases} H_{NC}(\hat{\vec{p}}, \hat{\vec{x}}) = \frac{1}{2m}\left[\left(\hat{p}_x - \frac{eB}{2c}\hat{y}\right)^2 + \left(\hat{p}_y + \frac{eB}{2c}\hat{x}\right)^2\right] + eE_x\hat{x} + eE_y\hat{y} \\ [\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{x}, \hat{y}] = i\hbar\theta \end{cases}$$

$$\hat{x} \to \hat{x} - \frac{\theta\hat{p}_y}{2}, \quad \hat{y} \to \hat{y} + \frac{\theta\hat{p}_x}{2}$$

$$\begin{cases} H_\theta(\hat{\vec{p}}, \hat{\vec{x}}) = \frac{1}{2m}\left[\left(\gamma\hat{p}_x - \frac{eB}{2c}\hat{y}\right)^2 + \left(\gamma\hat{p}_y + \frac{eB}{2c}\hat{x}\right)^2\right] \\ \qquad + eE_x\left(\hat{x} - \frac{\theta}{2}\hat{p}_y\right) + eE_y\left(\hat{y} + \frac{\theta}{2}\hat{p}_x\right) \\ [\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{x}, \hat{y}] = 0, \end{cases} \qquad \gamma = 1 - \frac{e\theta B}{4c}$$

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H_{NC}(\hat{p}, \hat{x})\psi(x,t) = H(\hat{p}, x) * \psi(x,t)$$

$$= H\left(\hat{p}, x_i - \frac{\theta_{ij}}{2}\hat{p}_j\right)\psi(x,t)$$

$$(f * g)(x) = \exp\frac{i\theta^{ij}}{2}\partial_{x_i}\partial_{y_j} f(x)g(y)\Big|_{y=x}$$ Moyal product

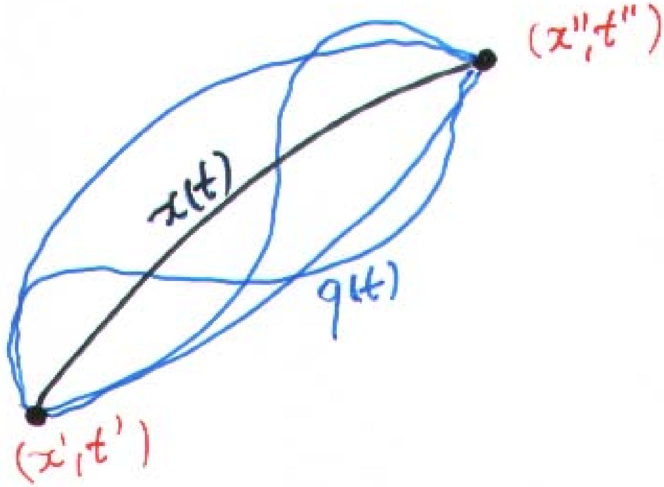
$[\hat{x}, \hat{y}] = i\hbar\theta$

$[\hat{x}, \hat{y}] = \hat{x}\hat{y} - \hat{y}\hat{x} = x * y - y * x = i\hbar\theta$

- Hall effect
- Ahoronov – Bohm effect (path integral)

$$B_{eff} = \frac{B}{1 - \frac{e\theta B}{hc}}$$ test NC

3. Path integral in ordinary QM (OQM)



$x' = q(t')$
$x'' = q(t'')$

$$\mathcal{K}(x'', t''; x', t') = \sum_{q(t)} \exp \frac{2\pi i}{h} S[q]$$

$$= \int_{(x', t')}^{(x'', t'')} \exp\left(\frac{2\pi i}{h} \int_{t'}^{t''} L(\dot{q}, q, t)\, dt\right) \mathcal{D}q$$

$$\psi(x'', t'') = \int \mathcal{K}(x'', t''; x', t')\, \psi(x', t')\, dx'$$

If $L(\dot{q}, q, t)$ is quadratic in $\dot{q}$ and q then

$$\boxed{\mathcal{K}(x'',t'';x',t') = \frac{1}{(ih)^{\frac{D}{2}}}\sqrt{\det\left(-\frac{\partial^2 \bar{S}}{\partial x''_k \partial x'_l}\right)}\ \exp \frac{2\pi i}{h} \bar{S}(x'',t'';x',t')}$$

$$\bar{S}(x'',t'';x',t') = \int_{t'}^{t''} L(\dot{x}, x, t)\, dt \qquad x(t) - \text{class. trajectory}$$

How to extend $\mathcal{K}(x'',t'';x',t')$ to NC case when $[\hat{x}_i, \hat{x}_j] = i\hbar\theta_{ij}$?

4. Path integral in NCQM

4.1. NC quadratic Lagrangians

D=2

- $$\begin{aligned} L(\dot{x},x,t) &= \alpha_{11}\dot{x}_1^2 + \alpha_{12}\dot{x}_1\dot{x}_2 + \alpha_{22}\dot{x}_2^2 \\ &+ \beta_{11}\dot{x}_1 x_1 + \beta_{12}\dot{x}_1 x_2 + \beta_{21}\dot{x}_2 x_1 + \beta_{22}\dot{x}_2 x_2 \\ &+ \gamma_{11} x_1^2 + \gamma_{12} x_1 x_2 + \gamma_{22} x_2^2 \\ &+ \delta_1 \dot{x}_1 + \delta_2 \dot{x}_2 + \eta_1 x_1 + \eta_2 x_2 + \phi \end{aligned}$$

$$\alpha_{ij} = \alpha_{ij}(t), \quad \beta_{ij} = \beta_{ij}(t), \ \ldots, \ \phi = \phi(t)$$

$$L(\dot{x},x,t) = \langle \alpha\dot{x},\dot{x}\rangle + \langle \beta x,\dot{x}\rangle + \langle \gamma x,x\rangle + \langle \delta,\dot{x}\rangle + \langle \eta,x\rangle + \phi$$

$$\alpha = \begin{pmatrix} \alpha_{11} & \frac{\alpha_{12}}{2} \\ \frac{\alpha_{12}}{2} & \alpha_{22} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_{11} & \frac{\gamma_{12}}{2} \\ \frac{\gamma_{12}}{2} & \gamma_{22} \end{pmatrix},$$

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

$\langle \cdot,\cdot \rangle$ = scalar product $\qquad \det\alpha \neq 0$

- $$p_j = \frac{\partial L}{\partial \dot{x}_j}, \qquad \dot{x} = \frac{1}{2}\alpha^{-1}(p - \beta x - \delta)$$

$$H(p,x,t) = \langle p,\dot{x}\rangle - L(\dot{x},x,t)$$

$$H(p,x,t) = \langle Ap,p\rangle + \langle Bx,p\rangle + \langle Cx,x\rangle + \langle D,p\rangle + \langle E,x\rangle + F$$

$$A = \frac{1}{4}\alpha^{-1}, \quad B = -\frac{1}{2}\alpha^{-1}\beta, \quad C = \frac{1}{4}\beta^T\alpha^{-1}\beta - \gamma,$$

$$D = -\frac{1}{2}\alpha^{-1}\delta, \quad E = \frac{1}{2}\beta^T\alpha^{-1}\delta - \eta, \quad F = \frac{1}{4}\delta^T\alpha^{-1}\delta - \phi$$

- $$H(p,x,t) \longrightarrow H(\hat{p},\hat{x},t)$$

$$[\hat{x}_k,\hat{p}_l] = i\hbar\delta_{kl}, \quad [\hat{x}_k,\hat{x}_l] = i\hbar\theta_{kl}$$

- $$\hat{x}_k = \hat{q}_k - \frac{\theta_{kj}}{2}\hat{p}_j, \quad \hat{p}_l = \hat{p}_l \Longrightarrow$$

$$[\hat{q}_k,\hat{q}_l] = 0, \quad [\hat{q}_k,\hat{p}_l] = i\hbar\delta_{kl}$$

$$H(\hat{p},\hat{x},t) \xrightarrow{\hat{x}=\hat{q}-\frac{1}{2}\Theta\hat{p}} H(\hat{p},\hat{q}-\tfrac{1}{2}\Theta\hat{p},t) =$$

$$= H_\theta(\hat{p},\hat{q},t) = \langle A_\theta\hat{p},\hat{p}\rangle + \langle B_\theta\hat{q},\hat{p}\rangle + \langle C_\theta\hat{q},\hat{q}\rangle$$

$$+ \langle D_\theta,\hat{p}\rangle + \langle E_\theta,\hat{q}\rangle + F_\theta$$

$$A_\theta = \left(A - \tfrac{1}{2}\Theta B - \tfrac{1}{4}\Theta C\Theta\right)_{sym}, \quad B_\theta = B + \Theta C$$

$$C_\theta = C, \quad D_\theta = D + \tfrac{1}{2}\Theta E, \quad E_\theta = E, \quad F_\theta = F$$

$$\Theta = \begin{pmatrix} 0 & \theta_{12} \\ -\theta_{12} & 0 \end{pmatrix}$$

- $H_\theta(\hat{p},\hat{q},t) \longrightarrow H_\theta(p,q,t)$

- $\dot{q}_k = \frac{\partial H_\theta}{\partial p_k}, \quad p = \frac{1}{2}A_\theta^{-1}(\dot{q} - B_\theta q - D_\theta)$

$$L_\theta(\dot{q},q,t) = \langle p,\dot{q}\rangle - H_\theta(p,q,t)$$

$$L_\theta(\dot{q},q,t) = \langle \alpha_\theta\dot{q},\dot{q}\rangle + \langle \beta_\theta q,\dot{q}\rangle + \langle \gamma_\theta q,q\rangle$$

$$+ \langle \delta_\theta,\dot{q}\rangle + \langle \eta_\theta,q\rangle + \phi_\theta$$

$$\alpha_\theta = \Big[\alpha^{-1} - \tfrac{1}{2}(\beta^T\alpha^{-1}\Theta - \Theta\alpha^{-1}\beta) + \Theta\gamma\Theta$$

$$- \tfrac{1}{4}\Theta\beta^T\alpha^{-1}\beta\Theta\Big]^{-1}$$

$$\beta_\theta = \alpha_\theta\left(\alpha^{-1}\beta - \tfrac{1}{2}\Theta\beta^T\alpha^{-1}\beta + 2\Theta\gamma\right)$$

$$\gamma_\theta = \frac{1}{4}\left(\beta^T\alpha^{-1} + \frac{1}{2}\beta^T\alpha^{-1}\beta\Theta - 2\gamma\Theta\right)\alpha_\theta\left(\alpha^{-1}\beta - \frac{1}{2}\Theta\beta^T\alpha^{-1}\beta + 2\Theta\gamma\right) - \frac{1}{4}\beta^T\alpha^{-1}\beta + \gamma$$

$$\delta_\theta = \alpha_\theta\left(\alpha^{-1}\delta - \frac{1}{2}\Theta\beta^T\alpha^{-1}\delta + \Theta\eta\right)$$

$$\eta_\theta = \frac{1}{2}\left(\beta^T\alpha^{-1} + \frac{1}{2}\beta^T\alpha^{-1}\beta\Theta - 2\gamma\Theta\right)\alpha_\theta\left(\alpha^{-1}\delta - \frac{1}{2}\Theta\beta^T\alpha^{-1}\delta + \Theta\eta\right) - \frac{1}{2}\beta^T\alpha^{-1}\delta + \eta$$

$$\phi_\theta = \frac{1}{4}\left\langle \delta_\theta, \alpha^{-1}\delta - \frac{1}{2}\Theta\beta^T\alpha^{-1}\delta + \Theta\eta \right\rangle - \frac{1}{4}\left\langle \alpha^{-1}\delta, \delta \right\rangle + \phi$$

$$\boxed{K_\theta(x'',t'';x',t') = \frac{1}{(ih)^{\frac{n}{2}}}\sqrt{\det\left(-\frac{\partial^2 S_\theta}{\partial x''_k \partial x'_l}\right)}\, \exp \frac{2\pi i}{h} S_\theta(x'',t'';x',t')}$$

$$S_\theta(x'',t'';x',t') = \int_{t'}^{t''} L_\theta(\dot{x}, x, t)\, dt$$

$x = x(t)$
class. trajectory

4.2. Some examples

Particle in a constant field in NC plane

$$L(\dot{x},x) = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \eta_1 x_1 - \eta_2 x_2$$

$$\alpha = \frac{m}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0, \quad \eta = -\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad \phi = 0$$

$$\alpha_\theta = \frac{m}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta_\theta = 0, \quad \gamma_\theta = 0, \quad \delta_\theta = \frac{m\theta}{2}\begin{pmatrix} -\eta_2 \\ \eta_1 \end{pmatrix}$$

$$\eta_\theta = -\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad \phi_\theta = \frac{m\theta^2}{8}(\eta_1^2 + \eta_2^2)$$

$$L_\theta(\dot{q},q) = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{m\theta}{2}(-\eta_2 \dot{q}_1 + \eta_1 \dot{q}_2)$$
$$- \eta_1 q_1 - \eta_2 q_2 + \frac{m\theta^2}{8}(\eta_1^2 + \eta_2^2)$$

$$m\ddot{q}_i = -\eta_i, \quad x_i(0) = x_i', \quad x_i(T) = x_i''$$

$$x_i(t) = x_i' - \frac{\eta_i t^2}{2m} + t\left(\frac{x_i'' - x_i'}{T} - \frac{\eta_i T}{2m}\right)$$

$$\bar{S}_\theta(x'', T; x', 0) = \int_0^T L_\theta(\dot{x},x)\,dt = \frac{m}{2T}\Big[(x_1'' - x_1')^2 +$$
$$+ (x_2'' - x_2')^2\Big] - \frac{T}{2}\Big[\eta_1(x_1'' + x_1') + \eta_2(x_2'' + x_2')\Big]$$
$$+ \frac{m\theta}{2}\Big[\eta_1(x_2'' - x_2') - \eta_2(x_1'' - x_1')\Big] - \frac{T^3}{24m}(\eta_1^2 + \eta_2^2)$$
$$- \frac{mT\theta}{8}(\eta_1^2 + \eta_2^2)$$

$$\mathcal{K}_\theta(x'',T;x',0) = \frac{1}{ih}\frac{m}{T}\exp\frac{2\pi i}{h}\bar{S}_\theta(x'',T;x',0)$$

$$= \mathcal{K}(x'',T;x',0)\exp\frac{2\pi i}{h}\Big\{\frac{m\theta}{2}\big[\eta_1(x_2''-x_2')-\eta_2(x_1''-x_1')\big] - \frac{mT\theta}{8}(\eta_1^2+\eta_2^2)\Big\}$$

Harmonic oscillator in NC plane

$$L(\dot{x},x) = \frac{m}{2}(\dot{x}_1^2+\dot{x}_2^2) - \frac{m\omega^2}{2}(x_1^2+x_2^2)$$

$$\alpha = \frac{m}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \beta = 0, \quad \gamma = -\frac{m\omega^2}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\delta = \eta = \phi = 0, \quad \Theta = \theta\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$\alpha_\theta = \frac{m}{2}\frac{1}{\mu}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \beta_\theta = \frac{m^2\omega^2\theta}{2\mu}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$\gamma_\theta = -\frac{m\omega^2}{2}\frac{1}{\mu}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \delta_\theta = \eta_\theta = \phi_\theta = 0, \qquad \mu = 1+\frac{\theta^2m^2\omega^2}{4}$$

$$L_\theta(\dot{q},q) = \frac{m}{2\mu}(\dot{q}_1^2+\dot{q}_2^2) + \frac{m^2\omega^2\theta}{2\mu}(-q_2\dot{q}_1+q_1\dot{q}_2) - \frac{m\omega^2}{2\mu}(q_1^2+q_2^2)$$

$$\ddot{q}_1 - m\omega^2\theta\dot{q}_2 + \omega^2 q_1 = 0$$
$$\ddot{q}_2 + m\omega^2\theta\dot{q}_1 + \omega^2 q_2 = 0$$

5. Conclusion

- Quadratic Lagrangians (Hamiltonians) for "NC classical mechanics" are found. NC classical dynamics can be developed along standard formalisms of classical and quantum mechanics.

$$
\begin{tikzcd}[column sep=large, row sep=huge]
L(\dot{x},x,t) \arrow[r] & H(p,x,t) \arrow[r] & H(\hat{p},\hat{x},t) \; [\hat{x}_k,\hat{x}_l]=i\hbar\theta_{kl} \arrow[d, "\theta:\ \hat{x}_k=\hat{q}_k-\frac{\theta_{kl}\hat{p}_l}{2}"] \\
L_\theta(\dot{q},q,t) \arrow[u, "\theta\to 0"] & H_\theta(p,q,t) \arrow[l] & H_\theta(\hat{p},\hat{q},t) \arrow[l]
\end{tikzcd}
$$

- Feynman's path integral for "NC quadratic Lagrangians" is formulated.

4. Path integrals in p-adic QM

$$\Delta x \geq l_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}\,\mathrm{cm}$$

Restriction on application of real numbers and archimedean geometry at Planck scale

$na \geq b$

more general approach (adelic approach):

- real + p-adic numbers
- archimedean + nonarchimedean geom.

– p-adic numbers	Kurt Hensel (1900)
– adeles	C. Chevalley (~1935)
– p-adic strings	I. Volovich (1987)
– p-adic QM	Vladimirov-Volovich (1988)
– adelic QM	B. Dragovich (1994)

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dragovich math-ph/... , hep-th/...
math-ph/0306023

watkins http://...
http://www.maths.ex.ac.uk/~mwatkins/zeta/physics7.htm

p-ADIC NUMBERS AND ADELES

$\mathbb{Q}$ = field of rational numbers

$x \in \mathbb{Q}$

REAL	p-ADIC
$x = \pm \sum_{n=n_0}^{-\infty} a_n 10^n$	$x = \sum_{m=m_0}^{+\infty} b_m p^m$
$a_n = 0, 1, \cdots, 9$	$b_m = 0, 1, \cdots, p-1$
ordinary absolute value	p-adic abs. value (p-adic norm)
$\lvert x \rvert_\infty$	$\lvert x \rvert_p$
$d_\infty(x,y) = \lvert x-y \rvert_\infty$	$d_p(x,y) = \lvert x-y \rvert_p$
$\lvert x \rvert_\infty = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$	$x = \frac{m}{n} = p^\nu \frac{a}{b}$ $\lvert x \rvert_p = p^{-\nu}, \quad \lvert 0 \rvert_p = 0$

$$\boxed{\lvert x \rvert_\infty \prod_p \lvert x \rvert_p = 1} \qquad x \in \mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$$

$$\lvert x+y \rvert_p \leq \max\{\lvert x \rvert_p, \lvert y \rvert_p\} \leq \lvert x \rvert_p + \lvert y \rvert_p$$

strong triangle inequality

nonarchimedean (ultrametric) norm !

$| \; | : F \longrightarrow \mathbb{R}_+$

1. $|x| \geq 0$, $|x| = 0 \longleftrightarrow x = 0$
2. $|xy| = |x||y|$
3. $|x+y| \leq |x| + |y|$

$$|x+y|_\infty \leq |x|_\infty + |y|_\infty$$ archimedean norm

$$|x+y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$$

nonarchimedean (ultrametric) norm

Examples

$x = \frac{15}{2} = \frac{3 \cdot 5}{2}$

$\left|\frac{15}{2}\right|_2 = 2$, $\left|\frac{15}{2}\right|_3 = \frac{1}{3}$, $\left|\frac{15}{2}\right|_5 = \frac{1}{5}$,

$\left|\frac{15}{2}\right|_p = 1 \quad p \geq 7.$

$$\boxed{|m|_p \leq 1, \quad m \in \mathbb{Z}}$$

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$$
$$= \sum_{n=0}^{+\infty} (p-1)p^n$$

All valuations on $\mathbb{Q}$

$(\mathbb{Q}, |\cdot|_\infty)$

$(\mathbb{Q}, |\cdot|_2)$

$(\mathbb{Q}, |\cdot|_3)$

$\vdots$

$(\mathbb{Q}, |\cdot|_\infty)$

$(\mathbb{Q}, |\cdot|_p)$

$p = 2, 3, 5, 7, 11, \ldots$

Ostrowski theorem: Each non-trivial valuation on $\mathbb{Q}$ is equivalent either to $|\cdot|_\infty$ or $|\cdot|_p$.

Completions of $\mathbb{Q}$

$$\begin{tikzcd} & \mathbb{R} \equiv \mathbb{Q}_\infty \\ \mathbb{Q} \arrow[ur, dash, "|\cdot|_\infty"] \arrow[dr, dash, "|\cdot|_p"'] & \\ & \mathbb{Q}_p \end{tikzcd}$$

$p = 2, 3, 5, \ldots$

$x \in \mathbb{R}: \quad x = \pm(x_n 10^n + \cdots + x_0 + x_{-1} 10^{-1} + \cdots)$

$x \in \mathbb{Q}_p: \quad x = x_{-k} p^{-k} + \cdots + x_0 + x_1 p^1 + x_2 p^2 + \cdots$

expansion into oposite directions

$|x|_p = p^k$

REAL NUMBER

$$x = \pm \sum_{n=k}^{-\infty} a_n 10^n$$

$$= \pm(a_k 10^k + \cdots + a_0 + a_{-1} 10^{-1} + \cdots)$$

P-ADIC NUMBER

$$x = \sum_{n=-k}^{+\infty} b_n p^n$$

$$= b_{-k} p^{-k} + \cdots + b_0 + b_1 p + b_2 p^2 + \cdots$$

$$b_i = 0, 1, \cdots, p-1$$

$\mathbb{Q}_p$ = field of p-adic numbers

$\mathbb{Z}_p$ = ring of p-adic integers

$$\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : x = x_0 + x_1 p + \cdots \}$$

$\mathbb{Z}$ is dense in $\mathbb{Z}_p$!

$$|x+y|_\infty \leq |x|_\infty + |y|_\infty$$

Archimedean norm

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}$$

non-Archimedean (ultrametric) norm

Geometry

$d(x,y) = |x-y|$

Archimedean geometry

$d_\infty(x,y) = |x-y|_\infty$

non-Archimedean geometry

$d_p(x,y) = |x-y|_p$

$$\begin{tikzcd}
 & \mathbb{R} \equiv \mathbb{Q}_\infty \arrow[r, dash] & \mathbb{C} \\
\mathbb{Q} \arrow[ur, dash, "| \ |_\infty"] \arrow[dr, dash, "| \ |_p"'] & & \\
 & \mathbb{Q}_p \arrow[r, dash] & \mathbb{C}_p
\end{tikzcd}$$

$\mathbb{Q}_p$ = field of p-adic numbers

$p = 2, 3, 5, 7, 11, \ldots$

archimedean geom.

$d_\infty(x,y) = |x-y|_\infty$

Geometries

$d(x,y) = |x-y|$

$d_p(x,y) = |x-y|_p$

non-archimedean geom.

$$d_\infty(x,y) \leq d_\infty(x,z) + d_\infty(z,y)$$

$$d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\}$$

p-adic spaces

closed ball $B_a(r)$

$$B_a(r) = \{x \in \mathbb{Q}_p : |x-a|_p \leq r\}$$

open ball $B_a(r^-)$

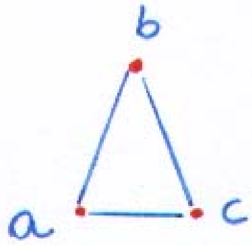
$$B_a(r^-) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}$$

sphere $S_a(r)$

$$S_a(r) = \{x \in \mathbb{Q}_p : |x-a|_p = r\}$$

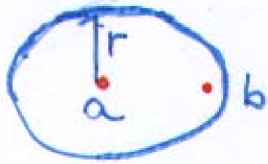
Some exotic properties of p-adic spaces

1) isosceles triangles



$$d_p(a,b) = d_p(b,c) \geq d_p(a,c)$$

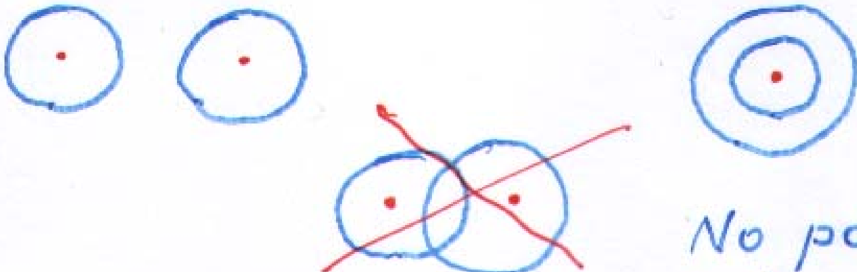
2)



$$B_a(r) = B_b(r)$$

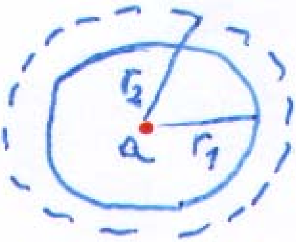
Each point of a ball may be its center!

3)



No partial intersection!

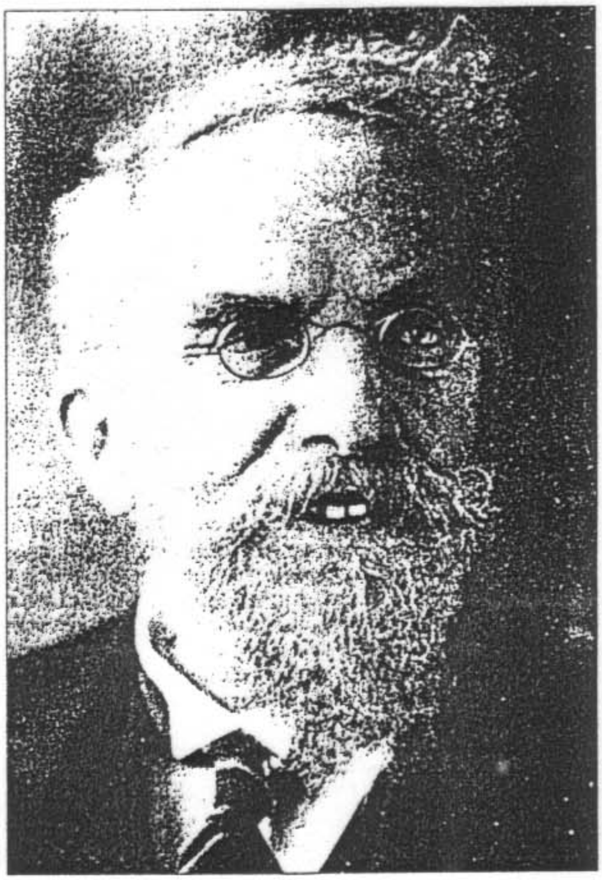
4)



$$|x-a|_p \leq p^{\nu} < p^{\nu+1}$$

closed = open

clopen sets



Le mathématicien
allemand ***Kurt Hensel***
(1861-1941)
inventa les nombres
p-adiques,
au début du XX^e siècle.
Il était un élève du
célèbre théoricien
des nombres
Leopold Kronecker.
Hensel enseigna à Berlin,
puis à l'université
de Marburg.
(Cliché Jean-Loup
Charmet)

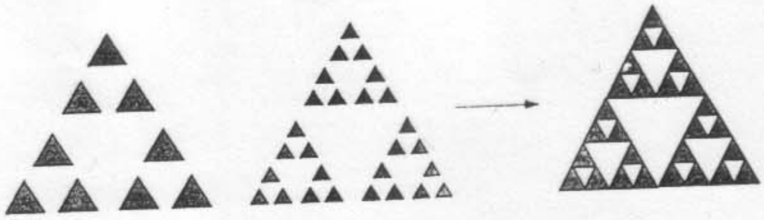


Figure 3: Model of $\mathbf{Z}_3$ and the Sierpinski gasket

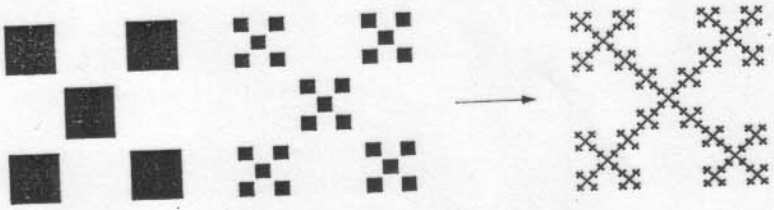


Figure 4: Model of $\mathbf{Z}_5$ and parametrization of a connected fractal

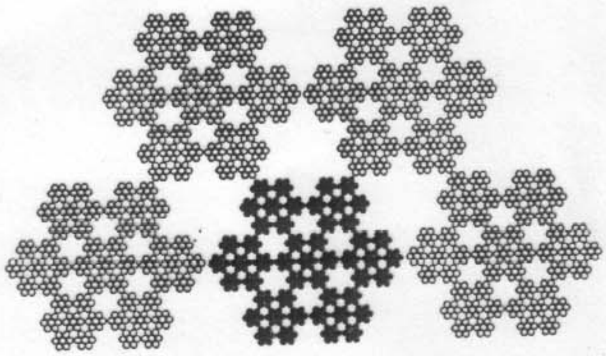


Figure 5: Model of a piece of Q_7

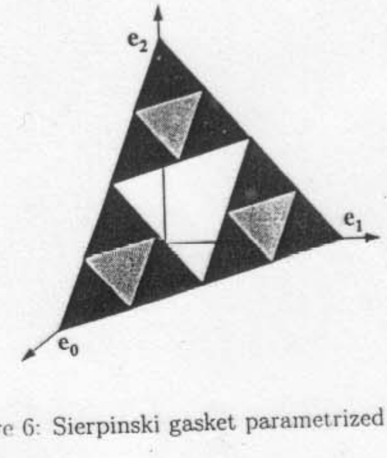


Figure 6: Sierpinski gasket parametrized by $\mathbf{Z}_3$

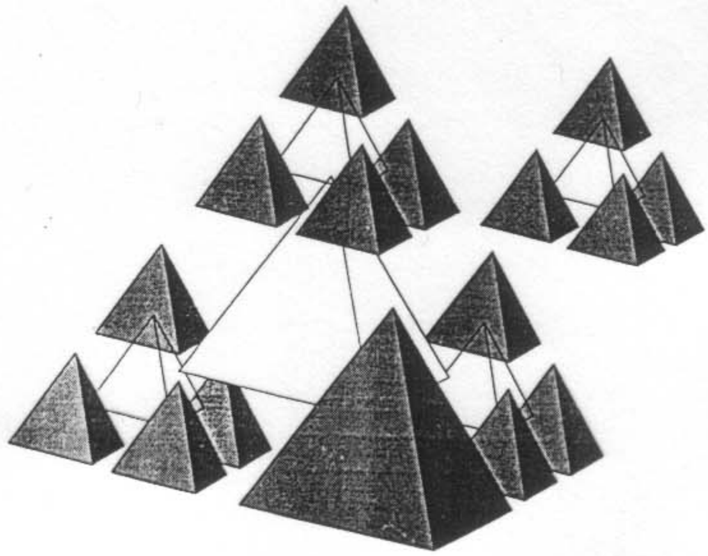


Figure 7: Space model of $\mathbf{Z}_5$

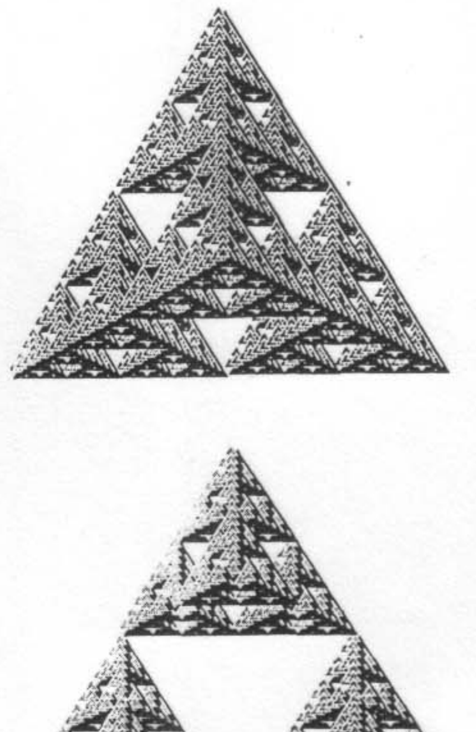


Figure 8: Top view of the space model of $\mathbf{Z}_5$

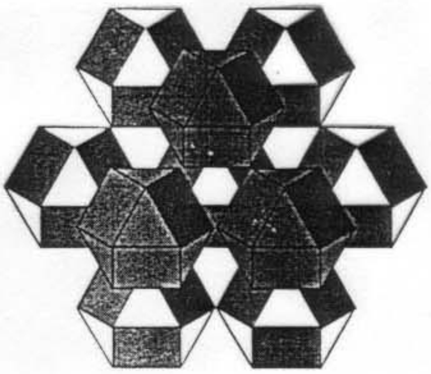


Figure 13: Model of $\mathbf{Z}_{13}$

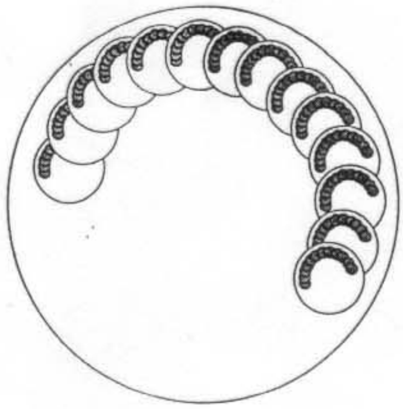
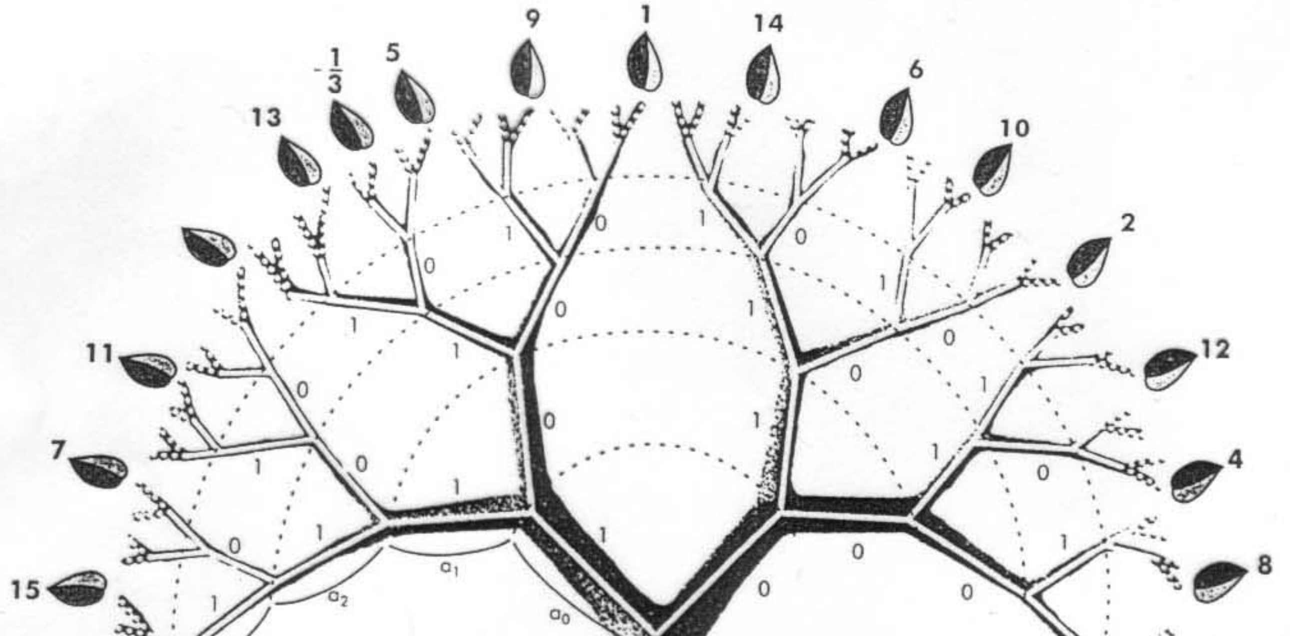


Figure 14: Antoine's model of Z_p ($p \succ 30$)



classical analysis

$\mathbb{R} \longrightarrow \mathbb{R}$ $\qquad$ $\mathbb{R} \longrightarrow \mathbb{C}$

classical theory $\qquad$ quantum theory

p-adic analysis

$\mathbb{Q}_p \longrightarrow \mathbb{Q}_p$ $\qquad$ $\mathbb{Q}_p \longrightarrow \mathbb{C}$

- $\pi_p(x) = |x|_p^a$ multiplicative character
- $\chi_p(x) = \exp(2\pi i \{x\}_p)$ additive character

$$\pi_\infty(x) \prod_p \pi_p(x) = 1 \qquad x \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$$

$$\chi_\infty(x) \prod_p \chi_p(x) = 1 \qquad x \in \mathbb{Q}$$

$$\chi_\infty(x) = \exp(-2\pi i x)$$

ADELES

$a = (a_\infty, a_2, a_3, \dots, a_p, \dots)$

$a_\infty \in R$, $a_p \in Q_p$, $p \in S$

$a_p \in Z_p$, $p \notin S$

S = finite set of p

$$\boxed{A = \bigcup_S A(S)} , \quad A(S) = R \times \prod_{p \in S} Q_p \times \prod_{p \notin S} Z_p .$$

A = topological ring of adeles

$Z_p = \{ x \in Q_p : |x|_p \leq 1 \}$

$r \in Q$

$a_r = (r, r, \dots, r, \dots)$ principal adele

ideles

$$A^\times = \bigcup_S A^\times(S) , \quad A^\times(S) = R^\times \times \prod_{p \in S} Q_p^\times \times \prod_{p \notin S} Z_p^\times$$

$Z_p^\times = \{ x \in Q_p : |x|_p = 1 \}$

$Q_p^\times = Q_p \setminus \{0\}$

$A^\times$ = multiplicative group of ideles

adelic analysis

1. $A \longrightarrow A$

2. $A \longrightarrow C$

motivations

- All experimental data belong $\mathbb{Q}$
- $\mathbb{Q}$ is dense in $\mathbb{R}$, but also in $\mathbb{Q}_p$
- There is plausible analysis on $\mathbb{Q}_p$ as well as on $\mathbb{R}$
- General mathematical methods and fundamental physical laws should be invariant under $\mathbb{R} \leftrightarrow \mathbb{Q}_p$
- Is there any aspect of the Universe that cannot be described without use of p-adic numbers?
- There is

$$\Delta x \gtrsim \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm$$

- Also Hasse-Minkowski (local-global) principle
- Adelic method is a natural approach to investigate p-adic (non-archimedean) effects in physics
- String/M-theory and quantum cosmology allow p-adic and adelic generalization

p-ADIC AND ADELIC QM

<u>p-adic QM</u> Vladimirov, Volovich

$$\left(L_2(\mathbb{Q}_p),\ W_p(z_p),\ U(t_p) \right)$$

- $L_2(\mathbb{Q}_p)$ Hilbert space on $\mathbb{Q}_p$
- $W_p(z_p)$ Weyl quantization on $L_2(\mathbb{Q}_p)$
- $U(t_p)$ unitary repr. of evolution oper. on $L_2(\mathbb{Q}_p)$

$$U_p(t)\psi(x) = \int_{\mathbb{Q}_p} \mathcal{K}_p(x,t;y,0)\psi(y)\,dy$$

$$\mathcal{K}_p(x'',t'';x',t') = \int_{\mathbb{Q}_p} \chi_p\left(-\int_{t'}^{t''} L(q,\dot q,t)\,dt\right) \prod_t dq(t)$$

quadratic Lagrangians Djordjevic, B.D.

$$\mathcal{K}_{\substack{p\\ \infty}}(x'',t'';x',t') = \lambda_{\substack{p\\ \infty}}\left(-\frac{1}{2}\frac{\partial^2 \bar S}{\partial x''\partial x'}\right) \left|\frac{\partial^2 \bar S}{\partial x''\partial x'}\right|_{\substack{p\\ \infty}}^{\frac{1}{2}} \chi_{\substack{p\\ \infty}}\left(-\bar S(x'',t'';x',t')\right)$$

number field invariant $(\mathbb{R} \leftrightarrow \mathbb{Q}_p)$

<u>adelic QM</u> B. D.

$$\left(L_2(\mathbb{A}),\ W(z),\ U(t) \right)$$

$$U(t)\psi(x) = \int_{\mathbb{A}} \mathcal{K}(x,t;y,0)\psi(y)\,dy$$

$$\mathcal{K}(x'',t'';x',t') = \mathcal{K}_\infty(x''_\infty,t''_\infty;x'_\infty,t'_\infty) \prod_p \mathcal{K}_p(x''_p,t''_p;x'_p,t'_p)$$

$$U(t)\psi(x) = \chi(Et)\psi(x)$$ spectral problem

5. Conclusion

- Path integral method has successful extension to QM on NC and p-adic spaces.
- For quadratic Lagrangians one can evaluate these path integrals analytically. Obtained expressions have the same form for real and p-adic cases.