Vortices and Seiberg-Witten Equations

(based on lectures at Nagoya University)

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(notes taken by Yuuji Tanaka)

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Contents

0.1 Preface

These notes are based on the lecture course, delivered in Nagoya University in November-December 2001 (from November 1 till December 10). The course included 9 two-hours lectures. The notes were taken by Yuuji Tanaka, he also produced the TeX-file of the notes. It was a hard work and I am very grateful to him for the cooperation.

I have visited Nagoya University by the invitation of Prof. Ryoichi Kobayashi, using a fellowship from JSPS (Japan Society for the Promotion of Science). I am very grateful to JSPS for awarding me with this fellowship and their help in the organization of my visit.

The fellowship was granted due to the invitation of Prof. Ryoichi Kobayashi. I've enjoyed his extremal hospitality during all the time when I stayed in Nagoya (not speaking of the long preparatory period and reports after the visit). Long discussions with Prof. Kobayashi have changed a lot my original vision of the subject and resulted in many changes in the text of the lectures. My sincere thanks to Ryoichi Kobayashi for all his efforts.

I am happy also to thank all my Japanese friends for their help and understanding.

0.2 Introduction

0.2.1 Self-duality equations

Occasionally, amazing things arise when geometers consider certain non-linear differential equations on manifolds, borrowed from physics. The moduli spaces of their solutions provide non-trivial invariants of the manifolds. A very successful example was brought by S.K.Donaldson [D] in the early 80's by using the self-duality equations, originating from the particle physics. Using the moduli spaces of these equations, he proved the diagonalizability of positive definite intersection forms of smooth simply connected four-manifolds. Later on, he had also introduced new invariants of such manifolds (called the Donaldson polynomials), produced from the moduli spaces of self-dual connections. The self-duality equations, as well as Donaldson polynomials, turned out to be a difficult object to study due to their non-Abelian nature and the non-compactness of the moduli spaces. There was an impression that both of these features are essential and inavoidable.

0.2.2 Seiberg-Witten equations

This impression was disproved in 1994, when N.Seiberg and E.Witten [?] have produced their equations, called now the Seiberg–Witten equations (or SW-equations, for brevity), which have no such drawbacks. They are essentially Abelian and have compact moduli spaces.

equations	gauge group	moduli space
self-duality	non-abelian, e.g. SU(2)	non-compact
Seiberg-Witten	abelian, i.e. U(1)	compact

Table 0.1: Comparison between self-dual and Seiberg-Witten equations

Moreover, they may be derived, as the self-duality equations, from a supersymmetric Yang–Mills theory in some limit (the self-duality equations correspond to the ultraviolet limit of the theory while the SW-equations — to the infrared one). Hence, one can expect, at least on the intuitive level, that any information, drawn from the self-duality equations, can be also derived from the SW-equations and with less efforts. The remarkable properties of the new equations have given rise to an unprecedented euphory among mathematicians, working in 4-dimensional topology, and inspired a huge amount of papers, dealing with SW-equations and their applications (cf., e.g. [SW1], [SW2], [W]). This enthusiasm turned out to be justified, since several difficult and well known mathematical problems, including the famous Thom conjecture (cf. [KM]) were quickly solved with the help of SW-invariants, produced from the moduli spaces of solutions of SW-equations.

0.2.3 Relation to Gromov invariants

Apart from Donaldson polynomials, the new SW-invariants of symplectic 4-manifolds are closely related to their Gromov invariant [Gr], counting the number of pseudoholomorphic curves in a given homology class. Namely,Taubes(cf.[T4],[T5],[T6], [T7], [T8]) has proposed an "equation"

$$\mathrm{Gr} = \mathrm{SW} \ ,$$

which is a mneumonic formula, expressing the existence of a simple relation between the Seiberg–Witten and Gromov invariants of symplectic 4-manifolds. The Taubes "equation" is based on a certain reduction procedure of SW-equations to pseudoholomorphic curves. The procedure (which is non-trivial and incorporates some limiting process) produces a family of the vortex equations (still another equation, coming from physics), defined on the normal bundle of the considered curve. The main goal of these lectures is to explain this reduction procedure, as well as its converse.

0.2.4 An outline of the lecture course

We start our long way to this goal from dimension 2 in Chapter 1, where we study the vortex equations on the complex plane and compact Riemann surfaces. We provide this Chapter with a physical introduction, explaining how these equations arise in the superconductivity theory. In Chapter 2 we switch to dimension 3, where the third variable can be considered as an extra space variable (the corresponding theory in this case describes the so called Abrikosov strings or vortex lines) or as the time variable (in that case we get a theory, describing the vortex dynamics). Chapter 3 contains a digression, devoted to Clifford algebras and spinor geometry. We end up in Chapter 4 in dimension 4, where we deal with the Seiberg-Witten equations on compact 4-manifolds. We explain here the Taubes' correspondence between solutions of Seiberg-Witten equations on symplectic 4-manifolds and pseudoholomorphic curves. Our exposition is based on the concept of the adiabatic limit, adopted in Chapter 2. We believe that this concept makes the whole procedure more physical and transparent.

Chapter 1

Dimension two — vortex equations

This chapter is devoted to the vortex equations, arising in the superconductivity theory. The necessary physical background from this theory is presented in Section 1.1 (a general reference for this Section is [LP]). In Section 1.2 we introduce the vortex equations on the complex plane. A complete description of the moduli spaces of their solutions is given by Taubes' theorems, presented in Section 1.3. In Section 1.4 we switch to the vortex equations on compact Riemann surfaces and study obstructions to their solvability (which do not exist in the complex plane case). The moduli space of vortex solutions is described by the Bradlow's theorem, proved in Section 1.5.

1.0.5 Superconductivity

The phenomena of superconductivity was first observed by Kamerlingh-Onnes in 1911, while he examined the resistance of mercury in low temperatures. It was known that the resistance of metals decrease when they are cooled down. Surprisingly, the resistance of mercury suddenly vanished at the temperature 4.15K. This phenomena was called the superconductivity (s-conductivity, for brevity) and it turned out later that many metals and alloys acquire this property for temperatures, close to the absolute zero.

Another characteristic feature of s-conductors, called the perfect diamagnetism, was discovered in 1933, when Meissner and Oschenfeld observed that the exterior magnetic field is pushed away from the s-conductor. This phenomena, called the Meissner effect, is now a practical criteria of s-conductivity. More precisely, the magnetic field $H(x)$ inside the s-conductor decays exponentially with the distance dist from the boundary

$$|H(x)| \leq Ce^{-\text{dist}/\delta},$$

where δ is called the penetration depth.

A theory, which describes the s-conductivity, was first proposed by London. He explained the Meissner effect by using the so called London equation. Next theoretical progress was made in 1950 when Ginzburg and Landau proposed their Lagrangian. Their theory gave a satisfactory explanation of the s-conductivity and, in particular, a derivation of the London equation. The modern microscopic s-conductivity theory, called BCS-theory after the names of its authors Bardeen, Cooper and Schrieffer, was created in 1957 and incorporated the macroscopic Ginzburg-Landau theory. According to BCS-theory, the s-conductivity phenomena is due to the formation of so called Cooper pairs inside the s-conductor under very low temperatures. These pairs are quasi-particles, formed by pairs of electrons; they have the double electron charge $e_* = 2e$ and zero spin (hence, opposite to electrons, these quasi-particles are bosons).

1.0.6 Two types of superconductors

If we increase the magnetic field outside the s-conductor, then for some critical value H_{cr} the s-conductivity breaks down and the magnetic field starts to penetrate inside the s-conductor. This process can proceed according to two different scenarios and, accordingly, all s-conductors are divided into two different classes. For s-conductors of the Ist type (which are mostly metals) it occurs as a sharp jump along the whole interior of s-conductor so that the graph of the magnetic field B inside the s-conductor with respect to the exterior magnetic field H has the form (cf. the left hand side of the Figure).

In other words, for $H = H_{\text{cr}}$ we have a phase transition of the Ist type.

For s-conductors of the IInd type (which are mostly alloys) the same process develops gradually, by small steps, so it may be considered physically as continuous. The graph of $B(H)$ will have the form (cf. the right hand side of the Figure). When the exterior magnetic field exceeds the first critical value H^1_{cr}, inside the s-conductor there appear certain tube zones of intermediate conductivity, called the flux tubes. In the centre of such a tube (cf. Fig. 1.2), along the so called Abrikosov string, the conductivity is already normal (n-conductivity) while outside the tube we still have the s-conductivity.

As the level of the exterior magnetic field increases, the number of flux tubes is also increased so that after the second critical value $H^2_{\rm cr}$ the tubes fill up the whole s-conductor, transforming it into a normal conductor.

1.0.7 Ginzburg-Landau Lagrangian

For the description of an s-conductor in the intermediate state Ginzburg and Landau proposed the following Lagrangian density

$$\mathcal{L} := \frac{\vec{B}^2}{8\pi} + \frac{\hbar^2}{m^*}\left|\left(\vec{\nabla} - \frac{ie^*}{\hbar c}\vec{A}\right)\Phi\right|^2 - \alpha|\Phi|^2 + \beta|\Phi|^4, \tag{1.0.1}$$

where $\vec{A}$ is the electromagnetic vector potential, $\vec{B} = \vec{\nabla} \times \vec{A}$ is the magnetic field (or magnetic induction), Φ is the wave function of a Cooper pair (or an order parameter), responsible for s-conductivity, $e^* = 2e$ and $m^* = 2m$ are respectively the charge and the mass of a Cooper pair, $\alpha, \beta > 0$ are physical parameters, $\hbar, c$ are the Planck constant and the light velocity respectively. In this Lagrangian density, the first term is the Lagrangian of magnetic field, the second term is the interaction term of magnetic field with Φ, written in the covariant form, and the last term is the self-interaction of Φ, which is responsible for non-linear character of Φ.

1.0.8 Gauge transformations

The Ginzburg–Landau Lagrangian density is invariant under *gauge transformations* of the following form:

$$\Phi \mapsto e^{-i\chi}\Phi \quad \text{(phase transformation)},$$

Figure 1.1: superconductivity of type I (left) and type II (right)

Figure 1.2: Abrikosov strings in type II superconductor

$$\vec{A} \mapsto \vec{A} - \frac{\hbar c}{e^*}\vec{\nabla}\chi \quad \text{(gradient transformation)},$$

where χ is a real valued function. If necessary, we can always get rid of this gauge freedom by fixing the gauge, for example, with the help of the following London gauge condition:

$$\vec{\nabla} \cdot \vec{A} = \mathrm{div}\vec{A} = 0.$$

1.0.9 Ginzburg–Landau equations

We introduce next the Ginzburg-Landau energy functional

$$E(\vec{A}, \Phi) := \int \mathcal{L}\, d\,\mathrm{vol}.$$

Then we obtain the Ginzburg-Landau equations as the Euler-Lagrange equations, associated to the first variation $\delta E(\vec{A}, \Phi) = 0$, as follows:

$$\frac{\hbar^2}{m^*}\left(\vec{\nabla} - \frac{ie^*}{\hbar c}\vec{A}\right)^2 + \alpha\Phi - \beta|\Phi|^2\Phi = 0, \tag{1.0.2}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mathrm{rot}\vec{B} = \frac{4\pi}{c}\vec{j}, \tag{1.0.3}$$

where $\vec{j}$ is the superconductivity current:

$$\vec{j} = \frac{e^*\hbar}{m^*c}\mathrm{Im}(\Phi\vec{\nabla}\Phi) - \frac{2e^{*2}}{m^*c}|\Phi|^2\vec{A}.$$

If we fix the gauge by the London gauge condition, then (1.0.3) becomes

$$\nabla^2\vec{A} - \frac{8\pi e^{*2}}{m^*c}|\Phi|^2\vec{A} = \frac{4\pi e^*\hbar}{m^*c}\mathrm{Im}(\Phi\vec{\nabla}\Phi). \tag{1.0.4}$$

We introduce now the dimensionless normalized density function of a Cooper pair by

$$\rho := \frac{\beta}{\alpha}|\Phi|^2,$$

and the penetration depth for $|\Phi|^2 = \rho\alpha/\beta$ by

$$\delta := \frac{c}{e^*}\sqrt{\frac{m^*\beta}{8\pi\alpha}}.$$

Then for $|\Phi| \equiv$ constant (corresponding to s-conductivity), (1.0.4) becomes

$$\Delta\vec{B} = \frac{\rho}{\delta^2}\vec{B}.$$

This is the London equation. From this equation, we can easily deduce the Meissner effect, namely, we have the following inequality

$$|\vec{B}(\vec{x})| \leq \text{const}\, e^{-\frac{\sqrt{\rho}}{\delta}\mathrm{dist}(\vec{x})}.$$

Exercise 1.0.1. Prove this estimate.

1.0.10 The dimensionless equations

In order to get rid of the coefficients which are not essential for our purposes, we introduce new variables:

$$\vec{x}' := \frac{\vec{x}}{\delta} \quad , \quad \Phi' := \sqrt{\frac{\beta}{\alpha}}\Phi \quad , \quad \vec{A}' := \frac{e^*\delta}{\hbar c}\vec{A} \quad , \quad \vec{B}' := \frac{e^*\delta^2}{\hbar c}\vec{B}.$$

Then we obtain the following dimensionless form of the Ginzburg-Landau Lagrangian density:

$$\mathcal{L} = \frac{\vec{B'}^2}{2} + |\nabla'_{\vec{A}}\Phi'|^2 + \frac{\mu^2}{2}(1 - |\Phi|^2)^2.$$

Figure 1.3: flux tube

Here we denote $\mu := \delta/\ell$. This is the only remaining physical constant. Practically, the penetration depth δ determines the characteristic size of $\vec{B}$, i.e. the rate of decaying of $\vec{B}$ inside the s-conductor, while $\ell := \hbar/\sqrt{m^*\alpha}$ determines the characteristic size of Φ.

In addition, we explain the physical meaning of μ. When $\mu < 1/\sqrt{2}$ (corresponding to $B \leq \Phi$), the Ginzburg-Landau theory describes the s-conductivity of the first kind. In the opposite case, for $\mu > 1/\sqrt{2}$ (corresponding to $B \geq \Phi$), it describes the s-conductivity of the second kind. In the second case, the flux tubes appear, i.e. the intermediate conductivity inside the tubes and s-conductivity outside them.

1.0.11 The structure of flux tubes

The magnetic flux through the flux tube is given by

$$\int_\sigma B d\sigma = \{\text{integer}\} \times \left\{ \pi \frac{\hbar e}{c} \right\},$$

where $\hbar e/c$ is a physical quantity called the flux quanta. In dimensionless units, this becomes

$$\int_\sigma B \, d\sigma = \{\text{integer}\} \times \pi.$$

In this case, $\vec{B}$ is directed along the string, and $\Phi = 0$ on the string. If we restrict Φ to σ and write $\Phi = \rho e^{i\theta}$, then $\vec{v} := \vec{\nabla}\theta$ will look like a hydrodynamic vortex. Therefore Abrikosov strings are also called vortex lines.

Finally, we explain the meaning of μ in terms of vortices. If $\mu > 1/\sqrt{2}$, i.e. we have the s-conductivity of the second kind, the vortices, rotating in the same direction, are repelled from each other. If $\mu < 1/\sqrt{2}$, the vortices, rotating in the same direction, attract each other. For $\mu = 1/\sqrt{2}$ we can expect that any collection of vortices should be realized.

1.1 Vortex equations

1.1.1 Two-dimensional reduction

We reduce the Ginzburg-Landau functional to that on the two-dimensional plane, orthogonal to $\vec{H}$, and suppose that time is fixed, i.e. we consider the static case. Denote by (x_1, x_2) the coordinates on this plane.

Then the Ginzburg-Landau Lagrangian density becomes

$$\mathcal{L}(A, \Phi) = |F_A|^2 + |d_A\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2, \tag{1.1.1}$$

where A is a U(1)-connection on $\mathbb{R}^2$, so we can write

$$A = A_1 dx_1 + A_2 dx_2,$$

where A_1, A_2 are smooth imaginary-valued functions. F_A is the curvature of A, i.e.

$$F_A = dA = \sum F_{ij} dx_i \wedge dx_j,$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ with $\partial_i := \partial/\partial x_i$. Also $d_A = d + A$ is the covariant exterior derivative, $\Phi = \Phi_1 + i\Phi_2$ is a complex-valued function, and $\lambda > 0$ is a constant ($\lambda = 1$ is called the critical value).

By using the Ginzburg-Landau density, we can define the potential energy:

$$U(A, \Phi) := \frac{1}{2}\int \mathcal{L}(A, \Phi) d^2x. \tag{1.1.2}$$

From the first variation $\delta U(A, \Phi) = 0$ we obtain the following Ginzburg-Landau equations:

$$\partial_i F_{ij} = J_i \qquad (j = 1, 2), \tag{1.1.3}$$

Figure 1.4: B and Φ around the Abrikosov string

Figure 1.5: hydrodynamic vortex

$$\nabla_A^2 \Phi = \frac{\lambda}{2}\Phi(|\Phi|^2 - 1), \tag{1.1.4}$$

where the current J is given by $J_j := \mathrm{Im}(\bar{\Phi}\nabla_{A,j}\Phi)$ with $\nabla_{A,j} := \partial_j + A_j$, and $\nabla_A^2 := \sum \nabla_{A,j}^2$.

1.1.2 Vortex number

If we suppose that $U(A, \Phi) < \infty$, then $\Phi \to 1$ for $|x| \to \infty$. Thus we can define the vortex number d as the winding number of the map

$$\Phi : S_R^1 \to \{|\Phi| \sim 1\} = S^1$$

for sufficiently large R. If $|d_A\Phi|$ decreases faster than $1/|x|^{1+\delta}$, then

$$d = \frac{i}{2\pi}\int F_A$$

holds, that is, d is interpreted as the total magnetic flux through the plane (x_1, x_2).

Exercise 1.1.1. Prove this.

1.1.3 Bogomol'nyi transformation

Now we introduce the vortices. They are minimizers of the potential energy $U(A, \Phi) < \infty$ for fixed d. We shall derive next the equations for them assuming that $\lambda = 1$.

Suppose that $d \geq 0$, and introduce a complex coordinate $z := x_1 + ix_2$, so that $\bar{\partial} := 1/2(\partial_1 + i\partial_2)$, $\bar{\partial}_A := \bar{\partial} + A^{0,1}$, where $A = A^{1,0} + A^{0,1}$ and

$A^{0,1} = -\bar{A}^{1,0}$. Then we can transform the potential energy $U(A,\Phi)$, using the *Bogomol'nyi transformation*:

$$U(A,\Phi) = \frac{1}{2}\int \underbrace{\left\{2|\bar{\partial}_A\Phi|^2 + \left|iF_{12} + \frac{1}{2}(|\Phi|^2 - 1)\right|^2\right\}}_{\text{sum of squares}} + \underbrace{\frac{i}{2}\int F_A}_{\text{topological}}.$$

In other words, $U(A,\Phi)$ is written as the sum of squares and the topological term (equal to πd).

Exercise 1.1.2. Prove it.

1.1.4 Vortex equations

This Bogomol'nyi formula implies a lower bound on $U(A,\Phi)$:

$$U(A,\Phi) \geq \pi d$$

for a fixed vortex number d, and the equality holds only for solutions of

$$\bar{\partial}_A\Phi = 0, \tag{1.1.5}$$

$$iF_{12} = \frac{1}{2}(1 - |\Phi|^2). \tag{1.1.6}$$

These equations are called the *vortex equations*. Note that the second equation is equivalent to

$$iF_A = *\frac{1}{2}(1 - |\Phi|^2). \tag{1.1.7}$$

For $d < 0$ there exists an analogous Bogomol'nyi transformation, which implies the following inequality:

$$U(A,\Phi) \geq -\pi d,$$

and the equality holds for solutions of

$$\partial_A\Phi = 0, \tag{1.1.8}$$

$$iF_{12} = \frac{1}{2}(|\Phi|^2 - 1). \tag{1.1.9}$$

These equations are called the *anti-vortex equations*.

1.2 Taubes' theorems

1.2.1 Formulations of theorems

The Taubes' theorems give a description of moduli spaces of solutions of vortex equations, i.e. the spaces of all solutions of these equations modulo gauge transformations. Hereafter, we call solutions of vortex equations the *vortex solutions* for short.

Recall that the vortex solutions are minimizers of the potential energy $U(A,\Phi)$ with $U(A,\Phi) < \infty$ for fixed $d > 0$, where A is an imaginary valued 1-form (U(1)-gauge potential) on $\mathbb{R}^2 \cong \mathbb{C}$, Φ is a complex valued function on $\mathbb{R}^2 \cong \mathbb{C}$, and d is the winding number of Φ at infinity. The vortex equations, derived in the previous subsection, have the form

$$\bar{\partial}_A \Phi = 0,$$

$$iF_{12} = \frac{1}{2}(1 - |\Phi|^2).$$

Note that gauge transformations of the form

$$A \mapsto A + id\chi \quad , \quad \Phi \mapsto e^{-i\chi}\Phi,$$

where χ is a real-valued function on $\mathbb{C}$, operate on the space of vortex solutions and the vortex equations are invariant under this action.

In [T1], Taubes proved the following

Theorem 1.2.1 (Taubes). *For $d \geq 0$ and any collection $\{z_1, z_2, \cdots z_k\}$ of different points in $\mathbb{C}$ with multiplicities $d_1, d_2, \cdots d_k$, such that $\sum d_j = d$, there exists a unique (up to gauge transformations) vortex solution (A,Φ) with $U(A,\Phi) < \infty$ subject to the condition*

$$\{zeros\ of\ \Phi\} = \sum d_j z_j.$$

It follows that the vortex number of (A,Φ) is equal to d. We call such a vortex solution the *d-vortex*. Note that an analogous theorem is true for $d < 0$ since any solution of the anti-vortex equations corresponds by the complex conjugation to a vortex solution. An analogue of d-vortex for the anti-vortex equations is called the *$|d|$-anti-vortex*.

Remark 1.2.2. For $d = 0$ any solution is gauge equivalent to the trivial one, i.e. $A \equiv 0, \Phi \equiv 1$.

Open question 1.2.3. It is unknown if the same is true for any (A, Φ) with $U(A, \Phi) < \infty$ and $d = 0$

Furthermore, in [T2], Taubes proved

Theorem 1.2.4 (Taubes). *Any critical point (A, Φ) of $U(A, \Phi) < \infty$ with $\lambda = 1$ and $d > 0$ or, equivalently, any solution of Euler-Lagrange equations with $U(A, \Phi) < \infty$, $\lambda = 1$ and $d > 0$ is gauge equivalent to some d-vortex solution, described in Theorem* 1.2.1.

Remark 1.2.5. Under the assumptions of Theorem 1.2.4, any solution of Ginzburg-Landau equations (1.2.3),(1.2.4) is either d-vortex or $|d|$-anti-vortex. In particular, there is no "vortex-anti-vortex" solution. This means physically that all solutions of Ginzburg-Landau equations are stable and have a minimal energy (in a given topological class).

Remark 1.2.6. The second order Ginzburg-Landau equations for critical points of $U(A, \Phi)$ are equivalent to the first order vortex equations for minima of $U(A, \Phi)$ under the assumptions of finite energy $U(A, \Phi) < \infty$ and $\lambda = 1$. This is a rare phenomena in gauge field theories, more often there exist non-minimal (or, physically, unstable) critical points of the action. This is the case for Bogomo'nyi-Prasad-Sommerfield (monopole) equations in $\mathbb{R}^3$, and for self-dual Yang-Mills equations in $\mathbb{R}^4$.

1.2.2 Vortex moduli space

We now introduce the vortex moduli space:

$$\mathfrak{M}_d := \frac{\{d\text{-vortices } (A, \Phi) \text{ with } d \geq 0\}}{\{\text{gauge transformations}\}}. \tag{1.2.1}$$

Then Theorem 1.2.1 and Theorem 1.2.4 imply

$$\mathfrak{M}_d = \mathrm{Sym}^d \mathbb{C}. \tag{1.2.2}$$

Note that $\mathrm{Sym}^d \mathbb{C}$ can be identified with $\mathbb{C}^d$ by assigning to a d-tuple of points in $\mathbb{C}$ a monic polynomial, having these points as its zeros.

1.2.3 Some estimates

We supply the above Theorems with the following important estimates.

Property 1.2.7. *For any d-vortex (A, Φ) with $d \geq 0$ and $U(A, \Phi) < \infty$ we have either $|\Phi(z)| \equiv 1$ or $|\Phi(z)| < 1$ for any $z \in \mathbb{C}$. Moreover,*

$$|d_A \Phi(z)| \leq C(1 - |\Phi|^2),$$

where $C > 0$ is some constant.

We know that $|\Phi| \to 1$ for $|z| \to \infty$. The latter estimate implies that $|\Phi| \to 1$ exponentially fast. The rate of convergence is determined by the constant C in this estimate. The next Property shows that this rate is determined by the characteristic size of Φ, i.e. by correlation radius ℓ.

Property 1.2.8. *For any solution (A, Φ) of Ginzburg-Landau equations (1.2.3),(1.2.4) with $U(A, \Phi) < \infty$ and $\lambda > 0$ and for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$*
such that

$$1 - |\Phi(z)|^2 \leq C_\varepsilon e^{-(1-\varepsilon)m_\Phi |z|},$$

$$|F_A(z)| \leq C_\varepsilon e^{-(1-\varepsilon)m_A |z|}$$

hold, where $m_\Phi \sim 1/\ell$ is the Cooper pair mass, and $m_A \sim 1/\delta$ is the mass of photon.

The proofs of the Properties 1.2.7 and 1.2.8 can be found in [JT].

1.2.4 The strategy of the proof of Theorem 1.2.3

In the rest of this Section we explain the idea of the proof of Theorem 1.2.3. We start with an approximate solution, satisfying the first vortex equation along with the boundary and divisor conditions. Then, plugging this into the second vortex equation, we obtain a non-linear elliptic equation for the error term, which is solved using a fixed point theorem.

In order to construct approximate solutions, we use the fact that for $|\Phi| \to 1$ the vortex equations become linear. So we take for an approximate solution a solution of these linear equations, given by the superposition of radial solutions with only one zero.

1.2.5 An ansatz

Consider an ansatz

$$\Phi = e^{(u+i\theta)/2},$$

where u and θ are real-valued functions. Since Φ has zeros at z_j, then $u(z) \to -\infty$ as $z \to z_j$. In addition, $\theta(z)$ is a multi-valued function with ramification points at z_j of order d_j.

The first vortex equation implies that $A^{0,1} = -\bar{\partial}\log\Phi$ holds outside zeros of Φ. As $A^{0,1}$ is smooth, this equality holds everywhere (here $\bar{\partial}\log\Phi$ should be interpreted as a current). Since $A^{1,0} = -\bar{A}^{0,1} = \partial\log\bar{\Phi}$, we have

$$A^{0,1} = -\bar{\partial}(u + i\theta) \quad , \quad A^{1,0} = \partial(u - i\theta).$$

Now we fix the gauge by choosing

$$\theta := \theta_0(z) = 2\sum_{j=1}^{k} d_j \mathrm{Arg}(z - z_j).$$

Then plugging this into the second vortex equation, we obtain

$$\Delta u = e^u - 1 + 4\pi\sum_{j=1}^{k}\delta(z - z_j). \tag{1.2.3}$$

1.2.6 Solving the Liouville-type equation

In order to solve this equation, we introduce a function

$$u_0(z) := -2\sum_{j=1}^{k}\log\left(1 + \frac{\mu}{|z - z_j|^2}\right)^{d_j}$$

with $\mu > 4d$. Note that the function u_0 satisfies the equation

$$\Delta u_0 = 4\pi\sum_{j=1}^{k} d_j\delta(z - z_j) - 4\sum_{j=1}^{k}\frac{\mu d_j}{(\mu + |z - z_j|^2)^2}.$$

Hence, setting $v := u - u_0$, we get the following equation for v:

$$\Delta v(z) = \underbrace{-1 + g(z)}_{f_1} + \underbrace{h(z)e^v}_{f_2}$$

with boundary condition: $v(z) \to 0$ as $|z| \to \infty$. Here

$$h(z) := e^{u_0(z)} \quad , \quad g(z) := 4\sum_{j=1}^{k}\frac{\mu d_j}{(\mu + |z - z_j|^2)^2},$$

where $0 < g(z) < 1$ since $\mu > 4d$.

According to the Kazdan-Warner theorem (cf. next Section), the equation

$$\Delta v = f_1 + f_2 e^v \quad (f_1 < 0,\ f_2 > 0)$$

with boundary condition $v(z) \to 0$ as $|z| \to \infty$ has a unique real analytic solution. Consequently, with the aid of this solution, we can construct the required d-vortex solution (A, Φ).

Note that a Liouville-type equation, as above, arises in differential geometry in the following problem: "For a given Riemannian metric g with Gaussian curvature κ, find a conformally equivalent Riemannian metric G with given Gaussian curvature K."

Setting $G = ge^{2v}$, we obtain the following Liouville-type equation for v:

$$-\Delta_g v = \kappa - Ke^{2v},$$

where Δ_g is the Laplace-Beltrami operator associated with g.

1.3 Vortex equations on compact Riemann surfaces

In this section we generalize the results of the previous section to compact Riemann surfaces.

1.3.1 Energy functional

Let X be a compact Riemann surface with Riemannian metric g and Kähler form ω. We fix a complex Hermitian line bundle $L \to X$ with Hermitian metric h and define the energy functional

$$U(A, \Phi) := \frac{1}{2} \int_X \left\{ |F_A|^2 + |d_A \Phi|^2 + \frac{1}{4}(1 - |\Phi|^2)^2 \right\} \omega. \tag{1.3.1}$$

Here A is a U(1)-connection on L, $F_A := dA$ is its curvature, d_A is the covariant exterior derivative generated by A, Φ is a section of $L \to X$, and $|\Phi| := ||\Phi||_h$. Note that this energy functional $U(A, \Phi)$ is invariant under gauge transformations given by $u \in \mathrm{Map}(X, U(1))$.

1.3.2 Bogomol'nyi transformation

The energy functional $U(A, \Phi)$ can be rewritten, using the Bogomol'nyi transformation, in the form

$$U(A, \Phi) = \int_X \left\{ |\bar{\partial}_A \Phi|^2 + \frac{1}{2} |iF_A^\omega + \frac{1}{2}(|\Phi|^2 - 1)|^2 \right\} \omega + \frac{i}{\pi} \int_X F_A, \quad (1.3.2)$$

where $F_A^\omega = \omega \lrcorner F_A (= (F_A, \omega))$ is the (1,1)-component of F_A, parallel to ω. This Bogomol'nyi formula follows from the relation

$$\int_X iF_A \Phi = - \int_X |\bar{\partial}_A \Phi|^2 \omega + \int_X |\partial_A \Phi|^2 \omega,$$

and the Kähler identities:

$$i[\omega \lrcorner, \bar{\partial}_A] = \partial_A^* \quad , -i[\omega \lrcorner, \partial_A] = \bar{\partial}_A^*.$$

According to the Gauss-Bonnet formula, the last term in Bogomol'nyi formula may be rewritten in the form

$$\frac{i}{\pi} \int_X F_A = 2c_1(L).$$

Consequently, if we suppose $c_1(L) > 0$, we obtain the lower bound for the energy:

$$U(A, \Phi) \geq \pi c_1(L),$$

and the equality is achieved only on solutions of the equations

$$\bar{\partial}_A \Phi = 0, \quad (1.3.3)$$

$$iF_A^\omega = \frac{1}{2}(1 - |\Phi|^2). \quad (1.3.4)$$

1.3.3 Necessary solvability condition

These equations look like the vortex equations on the complex plane. But in the case of a compact Riemann surface there is an evident obstruction to their solvability. Namely, integrating the second equation over X, we obtain

$$\frac{i}{2\pi} \int_X F_A = \frac{1}{4\pi} \int_X \omega - \frac{1}{4\pi} \int_X |\Phi|^2 \omega,$$

which can be rewritten as

$$c_1(L) = \frac{1}{4\pi}\mathrm{Vol}_g(X) - \frac{1}{4\pi}||\Phi||^2_{L^2}.$$

Thus we get a necessary condition for the solvability of the above equations:

$$c_1(L) \leq \frac{1}{4\pi}\mathrm{Vol}_g(X).$$

As we shall see next, this condition arises because of the non-invariance of energy under scale transformations.

1.3.4 Scale transformation

We introduce the scale transformation:

$$g_t := t^2 g \quad , \quad \omega_t := t^2\omega.$$

Under this scale transformation, the volume changes as

$$\mathrm{Vol}_{g_t}(X) = t^2\mathrm{Vol}_g(X).$$

Now the necessary solvability condition for the scaled metric g_t becomes

$$c_1(L) \leq \frac{t^2}{4\pi}\mathrm{Vol}_g(X).$$

This condition is satisfied for sufficiently large t. Hence, we can always satisfy the necessary solvability condition of the equations (2.2.3),(2.2.4) by scaling the original metric g.

1.3.5 Correct vortex equations

It is, however, more convenient to fix the metric and scale the definition of $U(A, \Phi)$ instead. Namely, we substitute the energy functional $U(A, \Phi)$ by its scaled version

$$U_\tau(A, \Phi) = \frac{1}{2}\int_X \left\{ |F_A|^2 + |d_A\Phi|^2 + \frac{1}{2}(\tau - |\Phi|^2)^2 \right\},$$

where $\tau > 0$ is the scaling factor.

Applying the Bogomol'nyi transformation to the scaled energy functional, we obtain the following lower bound:

$$U_\tau(A, \Phi) \geq \pi c_1(L),$$

where the equality is achieved only on solutions of the equations

$$\bar\partial_A \Phi = 0,$$

$$iF_A^\omega = \frac{1}{2}(\tau - |\Phi|^2).$$

These are correct vortex equations on compact Riemann surfaces. A necessary solvability condition for them has the form

$$c_1(L) \le \frac{\tau}{4\pi}\mathrm{Vol}_g(X).$$

1.4 Bradlow's theorem

In [B], Bradlow proved

Theorem 1.4.1 (Bradlow). *Let $d := c_1(L) > 0$ and D is an effective divisor on X of degree d, i.e. $D = \sum d_j z_j, \sum d_j = d$. Then the condition:*

$$c_1(L) < \frac{\tau}{4\pi}\mathrm{Vol}(X)$$

is necessary and sufficient for the existence of a unique (up to gauge) d-vortex solution (A, Φ) such that the zero divisor of $\Phi = D$.

Moreover, the holomorphic line bundle L with the holomorphic structure, given by $\bar\partial_A$, is isomorphic to $[D]$.

1.4.1 Reformulation

Note that the 1st vortex equation $\bar\partial_A \Phi = 0$ means that Φ is a holomorphic section of the Hermitian line bundle $(L, \bar\partial_A)$, where A is a Hermitian holomorphic connection on $(L, \bar\partial_A)$. Recall that such a connection is uniquely defined by the Hermitian metric.

We change now our point of view and fix a holomorphic structure on L, determined by a $\bar\partial$-operator $\bar\partial_L$, instead of the Hermitian metric. Given a holomorphic section Φ of $(L, \bar\partial_L)$, we shall look for a Hermitian metric H on L, such that the holomorphic connection A, compatible with H, satisfies the second vortex equation.

So, instead of the original problem:

Problem 1.4.2. Given a Hermitian line bundle (L, h), find a Hermitian connection A on L and a holomorphic section Φ of $(L, \bar\partial_A)$, satisfying the second vortex equation we consider the following

Problem 1.4.3. Given a Hermitian holomorphic line bundle $(L, h, \bar{\partial}_L)$ and a holomorphic section Φ of $(L, \bar{\partial}_L)$, find a Hermitian metric H on L, conformally equivalent to h, such that the connection A_H, compatible with H and $\bar{\partial}_L$, satisfies the second vortex equation.

1.4.2 Gauge action

On solutions of Problem 1.4.2, we have the action of the gauge transformation group $\mathcal{G} = \mathrm{Map}(X, U(1))$. On the other hand, there is a natural action of the complexified gauge transformation group $\mathcal{G}_{\mathbb{C}} = \mathrm{Map}(X, \mathbb{C}^*)$ on solutions of Problem 1.4.3. The latter action is given by gauge transformations of the form

$$\bar{\partial}_L \mapsto g(\bar{\partial}_L) = g \circ \bar{\partial}_L \circ g^{-1} \quad , \quad \Phi \mapsto g\Phi \quad , \quad H \mapsto |g^{-1}|^2 H$$

for $g \in \mathcal{G}_{\mathbb{C}}$.

Assertion 1.4.4. *There is a one-to-one correspondence between*

$$\{\textit{solutions } (A, \Phi) \textit{ of Problem } 1.4.2\}/\mathcal{G}$$

and

$$\{\textit{solutions } (\bar{\partial}_L, H, \Phi) \textit{ of Problem } 1.4.3\}/\mathcal{G}_{\mathbb{C}}$$

In order to obtain a solution of Problem 1.4.2 from that of Problem 1.4.3, we write $H = he^{2v} = hg^2$, and provide L with a new holomorphic structure

$$g(\bar{\partial}_L) = g \circ \bar{\partial}_L \circ g^{-1}.$$

Denote by A_g the connection on L, compatible with h and $g(\bar{\partial}_L)$, and by $\Phi_g := g\Phi$. Then (A_g, Φ_g) will be a solution of Problem 1.4.2.

1.4.3 Solution of Problem 1.4.3

Suppose that $(L, h, \bar{\partial}_L)$ is a holomorphic Hermitian line bundle together with a holomorphic section Φ. We are looking for a Hermitian metric $H = he^{2u}$ with $u \in \mathrm{Map}(X, \mathbb{R})$ such that

$$iF^{\omega}_{A_H} = \frac{1}{2}(\tau - |\Phi|^2_H)$$

for the holomorphic connection A_H, compatible with H. This equation is equivalent to the following Liouville-type equation for the conformal factor u :

$$-\Delta u = iF^{\omega}_{A_h} - \frac{\tau}{2} + \frac{1}{2}|\Phi|^2_h e^{2u},$$

where A_h is the connection, compatible with $\bar\partial_L$ and h. If we denote

$$f_1 := iF^{\omega}_A - \frac{\tau}{2} \quad , \quad f_2 := \frac{1}{2}|\Phi|^2_h,$$

then the latter equation becomes

$$-\Delta u = f_1 + f_2 e^{2u}. \tag{1.4.1}$$

Furthermore, we can get rid of one of the coefficients, if we put

$$c := 2\int_X f_1\omega = 2i\int_X F_A - \tau\int_X \omega = 4\pi c_1(L) - \tau \mathrm{Vol}(X). \tag{1.4.2}$$

Denoting by v a unique (up to a constant) solution of the Laplace equation:

$$-\Delta v = f_1 - \bar f_1,$$

with $\bar f_1 = \int_X f_1\omega$, we obtain for $w := 2(u-v)$ the following Liouville-type equation:

$$-\Delta w = c - fe^w,$$

where $f := -|\Phi|^2_h e^{2v}$ is a smooth non-positive function.

Now we use the Kazdan-Warner theorem [KW].

Theorem 1.4.5 (Kazdan-Warner). *Let X be a compact Riemann surface. Suppose that $f \in C^\infty(X,\mathbb{R})$ is not identically zero and $c \in \mathbb{R}$. Consider the Liouville-type equation:*

$$-\Delta w = c - fe^w \tag{1.4.3}$$

for $w \in C^\infty(X,\mathbb{R})$. Then

1. *If $c = 0$, then a solution of (1.4.3) exists if and only if $\bar f := \int_X f\omega < 0$ and $f > 0$ somewhere on X.*

2. *If $c < 0$, then*

   (a) *The condition* $\bar{f} < 0$ *is necessary for the solvability of* (1.4.3).

   (b) *Under the condition* $\bar{f} < 0$ *there exists a constant* $c_-(f)$ *with* $-\infty \leq c_-(f) < 0$ *such that a solution of* (1.4.3) *exists if and only if* $c > c_-(f)$.

   (c) *The equality* $c_-(f) = -\infty$ *holds if and only if* $f \leq 0$ *everywhere on* X. *In this case a solution of* (1.4.3) *is unique by the maximal principle.*

3. *If* $c > 0$, *then*

   (a) *The condition that* $f > 0$ *somewhere on* X *is necessary for the solvability of* (1.4.3).

   (b) *Under the necessary condition (a) there exists a constant* $c_+(f)$ *with* $0 < c_+(f) \leq +\infty$ *such that a solution of* (1.4.3) *exists if* $c > c_+(f)$.

Figure 1.6: the solvability diagram

In our case, $f \leq 0$ everywhere so $c < 0$ is a necessary and sufficient condition for the existence of a solution. Moreover, this solution is unique by the maximum principle. The inequality $c < 0$ is equivalent to the condition $4\pi c_1(L) < \tau \mathrm{Vol}\,(X)$, which is our hypothesis.

1.4.4 The end of the proof of Bradlow's theorem

Now we conclude the proof of Bradlow's theorem. For a given effective divisor D of degree d, we consider an associated holomorphic line bundle $(L, \bar{\partial}_L) = [D]$ and its canonical holomorphic section Φ such that the zero divisor of $\Phi = D$. Then the Kazdan-Warner theorem implies that there exists a unique Hermitian metric H, yielding a solution to Problem 1.4.3. And this is equivalent to the existence of a unique vortex solution $(\tilde{A}, \tilde{\Phi})$.

Exercise 1.4.6. Compute $(\tilde{A}, \tilde{\Phi})$ explicitly.

1.4.5 The critical case

Next we investigate the remaining critical case of the solvability condition, when

$$c_1(L) = \frac{\tau}{4\pi}\mathrm{Vol}\,(X).$$

By integrating the second vortex equation, we obtain

$$c_1(L) = \frac{\tau}{4\pi}\mathrm{Vol}\,(X) - \frac{1}{4\pi}||\Phi||^2$$

and it follows that $\Phi \equiv 0$.

We note again that the problem of solving the vortex equations (up to the gauge action of $\mathcal{G}$) is equivalent to the determination of a Hermitian metric H on a given holomorphic line bundle $(L, \bar{\partial}_L)$, which is conformally equivalent to h and satisfies the second vortex equation (up to the gauge action of $\mathcal{G}_{\mathbb{C}}$).

As $\tau = 4\pi c_1(L)/\mathrm{Vol}\,(X)$, $\Phi \equiv 0$, the second vortex equation becomes

$$iF^{\omega}_{A_H} = \frac{2\pi c_1(L)}{\mathrm{Vol}\,(X)}.$$

Note that this is an equation of the Einstein-Hermitian type. If we look for $H = he^{2u}$ for $u \in \mathrm{Map}(X, \mathbb{R})$, then u should satisfy the Laplace equation:

$$-\Delta u = iF^{\omega}_{A_h} - \frac{2\pi c_1(L)}{\mathrm{Vol}\,(X)},$$

where A_h is compatible with $\bar{\partial}_L$ and h. It has a unique solution (up to a constant).

Consequently, in the critical case, we have the one-to-one correspondence between

$$\{A = A_H \text{ on } (L, \bar{\partial}_L), \quad \text{satisfying the Einstein-Hermitian equations}\}/\mathcal{G}$$

and

$$\{\text{holomorphic line bundles } (L, \bar{\partial}_L)\}/\mathcal{G}_{\mathbb{C}} = \mathrm{Pic}\,(X).$$

Remark 1.4.7. 1. According to Bradlow's theorem, in the case:

$$c_1(L) < \frac{\tau}{4\pi}\,\mathrm{Vol}\,(X),$$

we have the one-to-one correspondence between

$$\{d\text{-vortex solutions } (A, \Phi)\}/\mathcal{G}$$

and

$$\{\text{effective divisors } D \text{ of } \deg d = c_1(L)\}.$$

Hence the moduli space of d-vortex solutions is equal to $\mathrm{Sym}^d X$.

2. The inequality

$$\tau > \frac{4\pi c_1(L)}{\mathrm{Vol}\,(X)}$$

coincides with the stability condition for the pair (E, Φ). Accordingly, the semi-stability condition for (E, Φ) is equivalent to the inequality

$$\tau \geq \frac{4\pi c_1(L)}{\mathrm{Vol}\,(X)}.$$

3. There is another proof of Bradlow's theorem by Garcia-Prada [Ga], based on the moment map argument.

Chapter 2

Dimension three — Abelian Higgs model

We switch our attention from the two-dimensional vortices, considered in the first Chapter, to the three-dimensional case. There are two possibilities to add the extra third variable. One is a Euclidian version, leading to the Ginzburg-Landau equations in $\mathbb{R}^3$, which describes the Abrikosov strings. The other is a Minkowski version, leading to the Ginzburg-Landau equations in $\mathbb{R}^{2+1}$, which describes the vortex dynamics in $\mathbb{R}^2$. In the first two Sections of this Chapter we deal with the vortex dynamics while the Euclidean model is considered in the last Section.

2.1 Adiabatic limit

2.1.1 Abelian (2+1)-dimensional Higgs model

We consider the following action functional:

$$S(A,\Phi) = \int \{T(A,\Phi) - U(A,\Phi)\},$$

where $U(A,\Phi)$ is given by the same formula as in $\mathbb{R}^2$, and $T(A,\Phi)$ is the kinetic energy:

$$T(A,\Phi) := \frac{1}{2}\int \{|d_{A,0}\Phi|^2 + |F_{0,1}|^2 + |F_{0,2}|^2\}.$$

In this formula $A = A_0 dt + A_1 dx_1 + A_2 dx_2$, where $A_\mu = A_\mu(t, x_1, x_2)$ $(\mu = 0, 1, 2)$ are smooth imaginary-valued functions on $\mathbb{R}^{2+1}$, and

$$F_A = dA = \sum_{\mu,\nu=0}^{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Th covariant derivative $d_A = d + A$ so that $d_{A,0}\Phi = \partial_t \Phi dt + A_0 \Phi dt$, where $\Phi = \Phi(t, x_1, x_2)$ is a complex-valued function on $\mathbb{R}^{2+1}$.

Taking the first variation $\delta S(A, \Phi) = 0$, we obtain the Euler-Lagrange equations as follows:

$$\partial_0 F_{0,j} + \sum_{k=1}^{2} \varepsilon_{jk} \partial_k F_{12} = i\mathrm{Im}(\bar{\Phi} \nabla_{A,j} \Phi) \quad (\text{for} \quad j = 1, 2), \tag{2.1.1}$$

$$(\nabla_{A,0}^2 - \nabla_{A,1}^2 - \nabla_{A,2}^2)\Phi = \frac{\lambda}{2} \Phi(1 - |\Phi|^2), \tag{2.1.2}$$

$$\partial_1 F_{01} + \partial_2 F_{0,2} = i\mathrm{Im}(\bar{\Phi} \nabla_{A,0} \Phi), \tag{2.1.3}$$

where $\nabla_{A,\mu} = \partial_\mu + A_\mu$ and $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$.

These equations are invariant under gauge transformations of the form

$$A \mapsto A + id\chi \quad , \quad \Phi \mapsto e^{-i\chi} \Phi,$$

where χ is a smooth real-valued function on $\mathbb{R}^{2+1}$.

2.1.2 Temporal gauge

We can choose the gauge so that $A_0 = 0$, it is called the temporal gauge. In this case the kinetic energy becomes

$$T(A, \Phi) = \frac{1}{2} \{ ||\dot{\Phi}||^2 + ||\dot{A}||^2 \},$$

where "dot" denotes the time derivative $\partial/\partial t = \partial/\partial x_0$, $||\cdot|| = ||\cdot||_{L^2(\mathbb{R})}$.

The Euler-Lagrange equations in the temporal gauge become

$$\ddot{A}_j + \sum \varepsilon_{jk} \partial_k F_{12} = i\mathrm{Im}(\bar{\Phi} \nabla_{A,j} \Phi) \quad \text{for } (j = 1, 2),$$

$$\ddot{\Phi} - \Delta_A \Phi = \frac{\lambda}{2} \Phi(1 - |\Phi|^2),$$

$$\partial_1 \dot{A}_1 + \partial_2 \dot{A}_2 = \mathrm{Im}(\bar{\Phi} \dot{\Phi}),$$

where $\Delta_A = \nabla_{A,1}^2 + \nabla_{A,2}^2$. Note that the latter equation is of initial condition type, i.e. it is satisfied for any $t > 0$, if it's true for $t = 0$.

2.1.3 Heuristic considerations

In this subsection we present an heuristic approach, due to Manton and Gibbons (cf. [Ma]), to solving approximately the above dynamic Euler-Lagrange equations.

A solution of the dynamic Euler-Lagrange equations (dynamic solution, for brevity) in the temporal gauge (modulo gauge transformations) may be considered as a smooth path

$$\gamma : t \longmapsto [A(t), \Phi(t)]$$

in the static configuration space:

$$\mathfrak{N}_d = \frac{\{\text{smooth data}\,(A,\Phi)\,\text{with}\,U(A,\Phi) < \infty\,\text{and vortex number}\ d\}}{\{\text{gauge transformations}\}}.$$

In other words, we interpret a dynamic solution as a 1-parameter vortex data on $\mathbb{R}^2$, depending on t, which is defined up to static gauge transformations (here and after $[A(t), \Phi(t)]$ denotes the gauge class of $(A(t), \Phi(t))$ with respect to static gauge transformations).

Suppose that $d > 0$, $\lambda = 1$ and define the kinetic energy of the path $\gamma(t) = [A(t), \Phi(t)]$ by

$$T(\gamma) := \frac{1}{2}\{||\dot{A}||^2 + ||\dot{\Phi}||^2\}.$$

Consider a family of paths γ_ε, depending on a parameter $\varepsilon > 0$ in such a way that $||T(\gamma_\varepsilon)|| \cong \varepsilon$. For small $\varepsilon > 0$ the paths γ_ε are close to the static moduli space $\mathfrak{M}_d$ and in the limit $\varepsilon \to 0$ they converge to a point in $\mathfrak{M}_d$.

However, if we introduce a slow time variable $\tau := \varepsilon t$ on γ_ε, then for $\varepsilon \to 0$ the paths γ_ε will tend to a path γ_0 on $\mathfrak{M}_d$, which is a geodesic of $\mathfrak{M}_d$ in T-metric. In other words, geodesics of $\mathfrak{M}_d$ in T-metric describe approximately the slowly moving dynamic solutions of the Euler-Lagrange equations for $\lambda = 1$.

This Manton-Gibbons' heuristic approach will be justified later in this Section after we introduce necessary mathematical tools.

2.1.4 Sobolev moduli spaces

In order to study the structure of the tangent space $T\mathfrak{M}_d$ of the vortex moduli space $\mathfrak{M}_d$, we introduce first a Sobolev version of $\mathfrak{M}_d$.

Denote by $\mathcal{V}^s := \mathcal{V}^s_d$ the space of d-vortex solutions of the vortex equations:

$$\bar{\partial}_A \Phi = 0,$$

Figure 2.1: vortex dynamics near the static vortex solutions

$$2idA = *(1 - |\Phi|^2),$$

where A is a 1-form with coefficients in the Sobolev space $H_s(\mathbb{C}, i\mathbb{R})$, $s \geq 1$, i.e.

$$A \in H_s(\mathbb{C}, i\mathbb{R}) \otimes \Omega^1(\mathbb{C}) =: \Omega^1_s(\mathbb{C}, i\mathbb{R}) =: \Omega^1_s,$$

and $\Phi \in H_s(\mathbb{C}, i\mathbb{R}) =: H_s$ so that $(A, \Phi) \in \Omega^1_s \times H_s$. We define

$$\mathcal{G}_s := \{\text{ gauge transformations, generated by } \chi \in H_s(\mathbb{C}, \mathbb{R})\}.$$

Then a Sobolev version of $\mathfrak{M}_d$ is defined by

$$\mathfrak{M}_d := \mathcal{V}^s_d / \mathcal{G}_{s+1}.$$

One can prove that $\mathfrak{M}^s_d = \mathrm{Sym}^d\mathbb{C}$, so $\mathfrak{M}^s_d$ does not depends on $s \geq 1$.

2.1.5 Linearized vortex equations

Varying the vortex equations with respect to A and Φ at some fixed solution (A, Φ) and dropping out the boundary terms, we obtain the linearized vortex equations:

$$\bar{\partial}_A \varphi + a^{0,1}\Phi = 0, \tag{2.1.4}$$

$$*(i(da)) + \mathrm{Re}(\varphi\bar{\Phi}) = 0, \tag{2.1.5}$$

where $(a, \varphi) \in \Omega^1_s \times H_s$.

Introduce the linearized vortex operator

$$\mathcal{D}_{A,\Phi} : \Omega^1_s \times H_s \to \Omega^{0,1}_{s-1} \times H_{s-1}(\mathbb{C}, \mathbb{R}),$$

defined by the left-hand-side of linearized vortex equations:

$$\mathcal{D}_{A,\Phi} : (a, \varphi) \longmapsto (\bar{\partial}_A \varphi + a^{0,1}\Phi, *i(da) + \mathrm{Re}(\varphi\bar{\Phi})).$$

Then we can define the tangent space of $\mathcal{V}^s_d$ at (A, Φ) by

$$T_{(A,\Phi)}\mathcal{V}^s_d = \ker \mathcal{D}_{(A,\Phi)} = \{(a, \varphi) \in \Omega^1_s \times H_s ; \mathcal{D}_{(A,\Phi)}(a, \varphi) = 0\}.$$

2.1.6 Infinitesimal gauge transformations

Note that the linearized vortex equations are invariant under infinitesimal gauge transformations given by

$$a \longmapsto a + id\chi \quad , \quad \varphi \longmapsto \varphi - i\Phi\chi$$

for $\chi \in H_{s+1}(\mathbb{C}, \mathbb{R})$. Then the orbit through the origin consists of $(id\chi, -i\Phi\chi)$. So we can define the tangential gauge operator:

$$\delta_{(A,\Phi)} : H_{s+1}(\mathbb{C}, \mathbb{R}) \to \Omega^1_s \times H_s(\mathbb{C}, \mathbb{C})$$

by

$$\chi \longmapsto (id\chi, -i\Phi\chi).$$

The adjoint operator

$$\delta^*_{(A,\Phi)} : \Omega^1_s \times H_s(\mathbb{C}, \mathbb{C}) \to H_{s-1}(\mathbb{C}, \mathbb{R})$$

is given by

$$(a, \varphi) \mapsto (d^*a + \mathrm{Im}(\bar{\Phi}\varphi)).$$

Since

$$\Omega^1_s \times H_s = T_{(A,\Phi)}(\mathcal{G}_{s+1}(A, \varphi) \oplus \ker \delta^*_{(A,\Phi)}),$$

we can fix the infinitesimal gauge by the gauge fixing condition:

$$\delta^*_{(A,\Phi)}(a, \varphi) = 0.$$

So the tangent space of $\mathfrak{M}^s_d$ can be given by

$$\begin{aligned} T_{(A,\Phi)}\mathfrak{M}^s_d &= \ker \mathcal{D}_{(A,\Phi)} \cap \ker \delta^*_{(A,\Phi)} \\ &= \{(a, \varphi) \in \Omega^1_s \times H_s ; \mathcal{D}_{(A,\Phi)}(a, \varphi) = \delta^*_{(A,\Phi)}(a, \varphi) = 0\}. \end{aligned}$$

Exercise 2.1.1. Prove that the restriction of $\mathcal{D}_{(A,\Phi)}$ to $\ker \delta^*_{(A,\Phi)}$ is a Fredholm operator with the index, equal to $2d$. Prove also that $\ker \mathcal{D}^*_{(A,\Phi)}|_{\ker \delta^*_{(A,\Phi)}} = 0$, which implies that $\ker \mathcal{D}_{(A,\Phi)}|_{\ker \delta^*_{(A,\Phi)}}$ is $2d$-dimensional.

2.1.7 Vortex paths

Consider again paths in the moduli space $\mathfrak{M}_d = \mathfrak{M}_d^s$ for some $s \geq 1$. We can describe such vortex paths, using the Taubes' theorem. Namely, by this theorem any path $t \mapsto q(t)$ in $\mathrm{Sym}^d\mathbb{C} \simeq \mathbb{C}^d$ uniquely determines a vortex path

$$\gamma : t \longmapsto [A(q(t)), \Phi(q(t))]$$

in $\mathfrak{M}_d$. We can also consider it as a path

$$\gamma : t \mapsto (A(q(t)), \Phi(q(t)))$$

in $\mathcal{V}_d$, satisfying the gauge fixing condition

$$\delta^*_{(A,\Phi)}(\dot{A}, \dot{\Phi}) = 0$$

for any t, where "dot" denotes, as before, the derivative by t.

2.1.8 Perturbations of vortex paths

Consider a perturbation $\tilde{\gamma}$ of the vortex path $\gamma = [A(q), \Phi(q)]$ in the configuration space $\mathfrak{N}_d$ of the form:

$$\tilde{\gamma}(t) = [\tilde{A}(t), \tilde{\varphi}(t)]$$

where

$$\tilde{A}(t) = A(q(t)) + a(t) \quad , \quad \tilde{\varphi}(t) = \Phi(q(t)) + \varphi(t).$$

We shall impose the following natural condition on (a, φ):

$$(a, \varphi) \perp T_{(A,\Phi)}\mathfrak{M}_d,$$

by which we exclude deformations in the directions, tangent to $T_{(A,\Phi)}\mathfrak{M}_d$.

We can obtain this orthogonality condition from the least squares method. Namely, given a path $\tilde{\gamma} = [\tilde{A}, \tilde{\Phi}]$ in $\mathfrak{N}_d$, which is supposed to be a dynamic solution, we are looking for a path $t \mapsto q(t)$ in $\mathrm{Sym}^d\mathbb{C}$ such that the corresponding d-vortex path $\gamma : t \mapsto [A(q(t)), \Phi(q(t))]$ is the "nearest" to $\tilde{\gamma}$. By the least squares method, such γ should minimize the functional

$$\frac{1}{2}\int \{||\tilde{A}(t) - A(q(t))||^2_{L^2} + ||\tilde{\varphi}(t) - \varphi(q(t))||^2_{L^2}\}.$$

The critical points of this functional satisfy the Euler-Lagrange equation

$$\langle a, \delta A\rangle + \langle \varphi, \delta\Phi\rangle = 0,$$

Figure 2.2: perturbation of vortex path

where $(\delta A, \delta\Phi)$ is a variation of $(A(q), \Phi(q))$ in q. So $(\delta A, \delta\Phi) \in T_{(A,\Phi)}\mathfrak{M}_d$, and

$$(a, \varphi) \perp T_{(A,\Phi)}\mathfrak{M}_d = \ker \mathcal{D}_{(A,\Phi)} \cap \ker \delta^*_{(A,\Phi)}.$$

Assuming the gauge fixing condition $\delta^*_{(A,\Phi)}(a, \varphi) = 0$, we obtain the above orthogonality condition

$$(a, \varphi) \perp \ker \mathcal{D}_{(A,\Phi)}. \tag{2.1.6}$$

If we have an L^2-basis $\{n_\mu\}$ of $\ker \mathcal{D}_{(A,\Phi)}$, that is, a basis of solutions of

$$\mathcal{D}_{(A,\Phi)} n_\mu = 0 \quad (\mu = 0, 1, \cdots, 2d),$$

then (2.1.6) is equivalent to

$$\langle (a, \varphi), n_\mu \rangle = 0$$

for $\mu = 1, 2, \cdots, 2d$.

2.1.9 Adiabatic equations

We introduce now a small parameter ε into our considerations. More precisely, we are looking for a dynamic solution $\tilde{\gamma} = [\tilde{A}(t), \tilde{\Phi}(t)]$, given by the perturbation of the vortex path γ of the form:

$$\tilde{A}(t) = A(q(t)) + \varepsilon^2 a(t) \quad , \quad \tilde{\Phi}(t) = \Phi(q(t)) + \varepsilon^2 \varphi(t),$$

satisfying the gauge fixing condition. We introduce the "slow-time" variable $\tau := \varepsilon t$. Plugging $(\tilde{A}, \tilde{\Phi})$ into the Ginzburg-Landau equations, we obtain

$$\partial_t^2(a,\varphi) + \mathcal{D}^*_{(A,\Phi)}\mathcal{D}_{(A,\Phi)}(a,\varphi) = (-\partial_\tau^2 A, -\partial_\tau^2)\Phi) + \varepsilon j, \tag{2.1.7}$$

where j is the sum of non-linear terms of current type. In the derivation of the equation above, we have used the fact that if $(A(q(t)), \Phi(q(t)))$ satisfies the vortex equations for any t, then it also satisfies the Ginzburg-Landau equations for $\lambda = 1$.

On the other hand, differentiating $\langle (a,\varphi), n_\mu \rangle = 0$ twice by t, we obtain

$$\langle \partial_t^2(a,\varphi), n_\mu \rangle = -\langle (a,\varphi), \partial_t^2 n_\mu \rangle - 2\langle \partial_t(a,\varphi), \partial_t n_\mu \rangle.$$

The first term on the right-hand-side of the equation above has the order ε^2, while the second term is of order ε.

We use this equation in (2.1.7). By taking the inner product of (2.1.7) with n_μ, we obtain

$$\langle \mathcal{D}^*_{(A,\Phi)}\mathcal{D}_{(A,\Phi)}(a,\varphi), n_\mu \rangle = \langle \left(-\partial_\tau^2 A, -\partial_\tau^2 \Phi)\right), n_\mu \rangle + \varepsilon h.$$

Since $\mathcal{D}_{(A,\Phi)} n_\mu = 0$, we obtain for $\varepsilon \to 0$

$$\langle \left(-\partial_\tau^2 A, -\partial_\tau^2 \Phi)\right), n_\mu \rangle = 0$$

for $\mu = 1, 2, \cdots, 2d$. We call these equations the *adiabatic equations.*

2.1.10 Justification of Manton-Gibbons's approach

We will show that these equations coincide with the Euler geodesic equations on $\mathfrak{M}_d$ with T-metric, thus justifying the Manton-Gibbons's approach. (Note that the same argument justifies the Atiyah-Hitchin's method to describe the scattering of slowly moving monopoles [AH]).

Recall that geodesics γ of the kinetic energy T are extremals of the functional:

$$\int_\gamma T(A,\Phi)d\tau = \frac{1}{2}\int_\gamma \{||\dot{A}||^2 + ||\dot{\Phi}||^2\} d\tau$$

defined on paths $\gamma : \tau \to [A(\tau), \Phi(\tau)]$ in $\mathfrak{M}_d$. The Euler-Lagrange equation for this functional has the form

$$\int_\gamma \{\langle \dot{A}, \delta\dot{A} + \langle \dot{\Phi}, \delta\dot{\Phi}\rangle\} d\tau = -\int_\gamma \{\langle \ddot{A}, \delta A\rangle + \langle \ddot{\Phi}, \delta\Phi\rangle\} d\tau = 0.$$

Since $(\delta A, \delta\varphi) \in T_{(A,\Phi)}\mathfrak{M}_d$, and we assume the gauge fixing condition, then

$$\langle \partial_\tau^2(A,\Phi), n_\mu \rangle = 0$$

for $\mu = 1, 2, \cdots, 2d$. These are precisely the adiabatic equations.

2.1.11 Adiabatic paths

This deduction of the adiabatic equations for $\lambda = 1$ may be considered as a hint that the same idea should work for $\lambda \neq 1$. In other words, the adiabatic equations for $\lambda \neq 1$ may be derived as an extremality condition on the action functional, restricted to paths in $\mathfrak{M}_d$.

Indeed, let us call a vortex path $\tau \to [A(\tau), \Phi(\tau)]$ in $\mathfrak{M}_d$ *adiabatic* if it is an extremal of the action $S(A, \Phi)$, restricted to paths γ lying in $\mathfrak{M}_d$.

The action functional has the form

$$S(\gamma) = S(A, \Phi) = \int_\gamma \{T(A, \Phi) - U(A, \Phi)\} d\tau,$$

where

$$T(A, \Phi) = T(\gamma) = \frac{1}{2}\{||\dot{A}||^2 + ||\dot{\Phi}||^2\},$$

$$U(A, \Phi) = U(\gamma) = \frac{1}{2}\{||dA||^2 + ||d_A\Phi||^2 + \frac{\lambda}{4}||1 - |\Phi|^2||^2\}.$$

Then the first variations of T and U are given by

$$\delta T(A, \Phi) = -\langle \ddot{A}, \delta A\rangle - \langle \ddot{\Phi}, \delta\Phi\rangle$$

$$\delta U(A, \Phi) = -\langle d^* dA + i\mathrm{Im}(\bar{\Phi} d_A \Phi), \delta A\rangle - \langle d_A^* d_A \Phi - \frac{\lambda}{2}\Phi(1 - |\Phi|^2), \delta\Phi\rangle.$$

Since the pair (A, Φ) satisfies the vortex equations for any τ, it also satisfies the Euler-Lagrange equations for $\lambda = 1$. Thus $\delta S(A, \Phi) = 0$ is equivalent to

$$\left(-\ddot{A}, -\ddot{\Phi} + \frac{\lambda - 1}{2}\Phi(1 - |\Phi|^2)\right) \perp T_{(A,\Phi)}\mathfrak{M}_d.$$

This condition (under the gauge fixing condition) is equivalent in terms of an L^2-basis $\{n_\mu\}$ of $\ker \mathcal{D}_{(A,\Phi)}$ to the equations

$$\langle\left(-\ddot{A}, -\ddot{\Phi} + \frac{\lambda - 1}{2}\Phi(1 - |\Phi|^2)\right), n_\mu\rangle = 0, \quad \mu = 1, \ldots, 2d.$$

These are the *adiabatic equations* for $\lambda \neq 1$.

2.1.12 Adiabatic Hamiltonian equations and adiabatic principle

The adiabatic equations

$$\langle \partial_t^2 (A, \Phi), n_\mu\rangle = \frac{\lambda - 1}{2}\langle \Phi(1 - |\Phi|^2), n_\mu\rangle, \quad \mu = 1, 2, \cdots, 2d$$

have a Newtonian form, i.e. the left-hand-side of these equations may be considered as "acceleration times mass", while the right-hand-side as "force". This is an indication that these equations are, in fact, Hamiltonian equations on $T^*\mathfrak{M}_d$, governed by an adiabatic Hamiltonian:

$$H_{\text{ad}} = T_{\text{ad}} + U_{\text{ad}}.$$

We shall write down an explicit expression for this Hamiltonian H_{ad} in local coordinates on $T^*\mathfrak{M}_d$.

Let $\{q_\mu\}$ be local coordinates on $\mathfrak{M}_d$ in a neighborhood of $q = [A, \Phi] \in \mathfrak{M}_d$, and $\{\dot{q}_\mu\}$ are local coordinates on $T_q\mathfrak{M}_d$. Denote, as before, by $\{n_\mu\}$ a basis of solutions of $\mathcal{D}_{(A,\Phi)} n_\mu = 0$. Then the T-metric on $T_q\mathfrak{M}_d$ is defined by

$$T_q(\dot{q}, \dot{q}) := \sum_{\mu,\nu=1}^{2d} \langle n_\mu, n_\nu \rangle \dot{q}_\mu \dot{q}_\nu.$$

Let $\{p_\mu\}$ be the momenta, i.e. fiber coordinates on $T_q^*\mathfrak{M}_d$, given by the Legendre transform

$$p_\mu := \sum_{\mu=1}^{2d} \langle n_\mu, n_\nu \rangle \dot{q}_\mu.$$

We provide $T_q^*\mathfrak{M}_d$ with the dual metric

$$T_q(p, p) := T_q(\dot{q}, \dot{q}).$$

Then the adiabatic Hamiltonian is given by

$$H_{\text{ad}} := \frac{1}{2} T_q(p, p) + U_{\text{ad}}(q),$$

where

$$U_{\text{ad}}(q) := \frac{|\lambda - 1|}{8} \int (1 - |\Phi|^2)^2 d^2x.$$

The corresponding adiabatic Hamiltonian equations have the form

$$\frac{dp_\mu}{d\tau} = -\frac{\partial H_{\text{ad}}}{\partial q_\mu} \quad \text{(Newton law)},$$

$$\frac{dq_\mu}{d\tau} = \frac{\partial H_{\text{ad}}}{\partial p_\mu} \quad \text{(definition of momentum)}.$$

We are now ready to state the adiabatic principle. It says that "any solution of adiabatic Hamiltonian equations can be approximated with any given precision by a solution of dynamic Euler-Lagrange equations."

2.2 Vortex dynamics

We demonstrate in this Section how the adiabatic principle, formulated in the previous Section, can be applied for the description of vortex dynamics.

2.2.1 Scattering of vortices

We consider first the scattering problem for two vortices on $\mathbb{C}$ in the critical case $\lambda = 1$. In the adiabatic limit this problem is reduced to the description of geodesics on the moduli space of 2-vortex solutions

$$\mathfrak{M}_2 = \mathrm{Sym}^2\mathbb{C}$$

with the T-metric.

Natural coordinates on $\mathrm{Sym}^2\mathbb{C}$ are provided by the following identification of $\mathrm{Sym}^2\mathbb{C}$ with $\mathbb{C}^2$:

$$\mathrm{Sym}^2\mathbb{C} \ni (z_1, z_2) \longmapsto (z_1 + z_2, z_1 z_2) \in \mathbb{C}^2$$

In the centre-of-mass coordinates we have

$$z_1 + z_2 = 0 \quad , \quad z_1 z_2 = a^2$$

for some $a \in \mathbb{C}$. We are looking for a geodesic $[A(t), \Phi(t)]$ on $\mathfrak{M}_2$, written in the form

$$\Phi(z) = (z - a)(z + a) f(z)$$

where a and f depend on t and f satisfies the following conditions:

1. $f > 0$ everywhere on $\mathbb{C}$ (gauge fixing condition);
2. $|f(z)| \sim \frac{1}{|z|^2}$ for $|z| \to \infty$ (asymptotic condition).

The kinetic energy

$$T(A, \Phi) = \frac{1}{2} \int \left\{ |\dot{A}_1|^2 + |\dot{A}_2|^2 + |\dot{\Phi}|^2 \right\} d\,\mathrm{vol}$$

may be (after a tedious computation) written in the form

$$T = \frac{1}{2}(\dot{\rho}^2 m_{||} + \rho^2 \dot{\theta} m_{\perp})$$

where $a = \rho e^{i\theta}$ and

$$\begin{aligned} m_{||} &= m_{||}(\rho, \theta) = \int \left\{ 4\rho^2 f^2 + \tfrac{1}{4} \tfrac{\partial f^2}{\partial \rho} \tfrac{\partial g^2}{\partial \rho} \right\} d\,\mathrm{vol}, \\ m_{\perp} &= m_{\perp}(\rho, \theta) = \int \left\{ 4\rho^2 f^2 + \tfrac{1}{4\rho^2} \tfrac{\partial f^2}{\partial \theta} \tfrac{\partial g^2}{\partial \theta} \right\} d\,\mathrm{vol} \end{aligned}$$

with $g^2(z) = (z-a)^2(z+a)^2$.

Since the kinetic energy does not depend explicitly on t and φ in polar coordinates $z = re^{i\varphi}$, we have two integrals of the Euler-Lagrange equations for T, corresponding to the energy and orbital momentum conservation laws:

$$T =: c_T = \mathrm{const} \quad , \quad M = \rho^2 \dot{\theta} m_{\perp} =: c_M = \mathrm{const}.$$

From these conservation laws we draw an equation for the geodesic $\rho = \rho(\theta)$ with given constants c_T, c_M:

$$\theta = \int_{\infty}^{\rho(\theta)} \frac{\sqrt{m_{||}/m_{\perp}}\, d\rho}{\rho \sqrt{\frac{2c_T m_{\perp}}{c_M^2}\rho^2 - 1}}$$

with the asymptotic condition: $\rho(\theta) \to \infty$ for $\theta \to 0$.

Figure 2.3: scattering of two vortices

In particular, we can determine from this equation the main parameters, characterizing the trajectory $\rho = \rho(\theta)$, namely, the minimal distance from the origin ρ_{min} and the scattering angle $\Delta\theta$. For ρ_{min} we have the equation

$$\frac{d\rho}{d\theta}(\rho_{min}) = 0 \Longleftrightarrow \frac{2c_T}{c_M^2} m_{\perp}(\rho_{min}) = \frac{1}{\rho_{min}^2}.$$

The scattering angle is defined by:

$$\Delta\theta = 2\int_{\infty}^{\rho_{min}} \frac{\sqrt{m_{||}/m_{\perp}}\,d\rho}{\rho\sqrt{\frac{2c_T m_{\perp}}{c_M^2}\rho^2 - 1}}.$$

The most interesting limiting case corresponds to $\rho_{min} \to 0$. In this case the main contribution to the integral, defining the scattering angle, is given by the integration near $\rho \sim 0$. For small ρ we can use the expansion of f^2 into the power series in ρ^2:

$$f^2 = f_0^2(1 + \rho^2 f_1 + \rho^4 f_2 + \dots)$$

where f_0^2 is the radial solution with $\Phi_0(z) = z^2 f_0$ (for $a = 0$). In this case

$$m_{\perp} = \mu\rho^2 + O(\rho^6) \quad , \quad m_{||} = m_{\perp} + O(\rho^6)$$

so for small ρ we have

$$\theta(\rho) \sim \int_0^{\lambda(\theta)} \frac{d\lambda}{\sqrt{2\frac{2c_T\mu}{c_M^2\lambda^2} - \lambda^2}} = \frac{1}{2}\arcsin\frac{c_M^2\lambda^2(\theta)}{\sqrt{2c_T\mu}}$$

for $\lambda(\theta) = 1/\rho(\theta)$. It implies the equation

$$\rho^2\sin 2\theta = \frac{c_M}{\sqrt{2c_T\mu}} = \rho_{min}^2,$$

where the second equality follows from the defining equation for ρ_{min} above. So the graph of $a = a(t)$ is the hyperbola, given by the equation

$$\operatorname{Re} a \cdot \operatorname{Im} a = \frac{\rho_{min}^2}{2}$$

and the scattering angle is equal to

$$\Delta\theta = \frac{\pi}{2}.$$

One can investigate in a similar way another limiting case $\rho_{min} \to \infty$ and show that $\Delta\theta \to \pi$ in this limit. It means, in other words, that there is no far-distant force in our problem (cf. details in [Ch – S])).

2.2.2 Periodic vortices

As another example of applications of the adiabatic principle we describe a periodic 2-vortex solution on the Riemannian sphere $S^2 = \mathbb{C}P^1$, found by [S].

Consider the Abelian $(2+1)$-dimensional Higgs model on the manifold

$$X = \mathbb{R}_t \times S^2,$$

provided with the Lorentz metric $ds^2 = dt^2 - g$, where g is the standard Riemannian metric on the sphere S^2_R of radius R in $\mathbb{R}^3$. The action for this model is given by

$$S_{\lambda,\tau}(A,\Phi) = \int \{T(A,\Phi) - U_{\lambda,\tau}(A,\Phi)\}dt$$

where

$$\begin{array}{rcl} T(A,\Phi) & = & \frac{1}{2}\int_{S^2}\left\{|\dot{A} - dA_0|^2 + |\dot{\Phi} - A_0\Phi|^2\right\} d\,\mathrm{vol}, \\ U_{\lambda,\tau}(A,\Phi) & = & \frac{1}{2}\int_{S^2}\left\{|dA|^2 + |d_A\Phi|^2 + \frac{\lambda}{4}(\tau - |\Phi|^2)^2\right\} d\,\mathrm{vol}. \end{array}$$

Here A is a U(1)-connection on a Hermitian line bundle $L \to S^2$, provided with a Hermitian metric h, and d_A is the corresponding exterior covariant derivative.

We suppose that L is extended to a Hermitian line bundle $\mathcal{L} \to X = \mathbb{R} \times S^2$, provided with a U(1)-connection

$$\mathcal{A} = A_0 dt + A = A_0 dt + A_1 dx_1 + A_2 dx_2,$$

and Φ is a section of $\mathcal{L} \to X$. We also suppose that the necessary solvability condition for vortex equations on S^2 is satisfied, namely

$$\tau > 4\pi \frac{d}{\mathrm{Vol}(S^2)},$$

and consider dynamic solutions for τ, close to the critical value

$$\tau_{\mathrm{cr}} = \frac{4\pi d}{\mathrm{Vol}(S^2)}.$$

We introduce affine coordinates $x = (x_1, x_2)$ on $S^2_R \setminus \infty$, using the stereographic projection: $S^2_R \setminus \infty \to \mathbb{R}^2_{(x_1,x_2)}$, and identify $\mathbb{R}^2$ with $\mathbb{C}$, provided with the complex coordinate $z = x_1 + x_2$. Suppose that the Hermitian metric h on L, restricted to $\mathbb{C}$, is determined by a function $h(z)$ so that $|\Phi(z)|_h^2 = h(z)|\Phi|^2$. The stereographic metric on $\mathbb{R}^2_{(x_1,x_2)}$ has the form

$$d\,\mathrm{vol} = \Lambda^2 dx_1 dx_2 \quad \text{for} \quad \Lambda = \frac{4R^2}{(1+|x|^2)^2}.$$

The dynamic Euler-Lagrange equations for our action have the form

$$\partial^2_{t,A_0}\Phi - \frac{1}{h\Lambda^2}\sum_{j=1}^{2}\partial_{j,A_j}(h\partial_{j,A_j}\Phi) - \frac{\lambda}{2}\Phi(\tau - |\Phi|^2) = 0 \tag{2.2.1}$$

$$\ddot{A}_j + \partial_j\dot{A}_0 + \epsilon_{jk}\partial_k\left(\frac{F_{12}}{\Lambda^2}\right) = i\mathrm{Im}(\bar{\Phi}\partial_{j,A_j}\Phi) \quad ,j=1,2 \tag{2.2.2}$$

$$\partial_j\dot{A}_j - \Delta A_0 = i\Lambda^2\mathrm{Im}(\bar{\Phi}\partial_{t,A_0}\Phi) \tag{2.2.3}$$

Figure 2.4: periodic vortex-vortex solution on sphere

In the adiabatic limit these equations are reduced to Hamiltonian equations on the moduli space of 2-vortices

$$\mathfrak{M}_2 = \mathrm{Sym}^2 S^2 \cong \mathbb{CP}^2,$$

governed by the adiabatic Hamiltonian

$$H_{\mathrm{ad}} = T_{\mathrm{ad}} + U_{\mathrm{ad}}.$$

To describe this Hamiltonian more explicitly, we consider the affine part $\mathbb{C}^2$ of $\mathfrak{M}_2$ with coordinates (z_1, z_2), assuming that zeros of Φ belong to $\mathbb{C}^2$, and introduce the centre-of-mass coordinates

$$z_1 + z_2 = 0 \quad , \quad z_1 z_2 = a^2$$

for $a \in \mathbb{C}$, written in the polar form as $-a^2 = \rho e^{i\theta}$. We make use of a small parameter $\delta > 0$, defined by

$$\delta^2 = 4\pi(\tau R^2 - d)$$

where $d = 2$.

In these coordinates

$$T_{\text{ad}} = \frac{1}{2} F(\rho)(\dot{\rho}^2 + \rho^2 \dot{\theta}^2)$$

with

$$F(\rho) = 2\delta^2 \frac{\rho^2 + 4\rho + 1}{(1+\rho)^2(1+\rho^2)^2} + O(\delta^4).$$

The potential energy is given by

$$U_{\text{ad}} = \frac{|\lambda - 1|}{8} \int_{S^2} (\tau - |\Phi|^2)^2 d\,\text{vol}$$

and depends only on ρ (i.e. the distance between vortices). It has the following power series decomposition with respect to the small parameter δ:

$$U_{\text{ad}} = \frac{|\lambda - 1|}{8} \left(4\pi\tau d - \tau\delta^2 + \frac{3\delta^4}{20\pi R^2} + \dots \right).$$

Denote by $r(\theta)$ the rotation of $\mathbb{C}$ by the angle θ: $z \mapsto e^{i\theta} z$, and by $r_*(\theta)$ the induced action of $r(\theta)$ on the configuration (A, Φ). (Note that the pullback of $r(\theta)$ to L is defined up to gauge transformations, so we should fix some pullback of $r(\theta)$ to L). We call by the periodic trajectory (of frequency ω) in the space $\mathcal{V}_d$ of d-vortex solutions any path $t \mapsto (\tilde{A}(t), \tilde{\Phi}(t))$ in $\mathcal{V}_d$ of the form

$$\tilde{A}(t) = r_*(\omega t) A + i d\chi \quad , \quad \tilde{\Phi}(t) = r_*(\omega t) \Phi e^{i\chi},$$

where $\chi = \chi(t, x)$ is obtained from a real-valued function $\chi_0(x)$ by the averaging the circle action

$$\chi(t, x) = \int_0^{\omega t} \chi_0(e^{i\omega s} x) ds,$$

satisfying the gauge fixing condition:

$$\delta^*_{(\tilde{A}, \tilde{\Phi})}(\partial_t \tilde{A}, \partial_t \tilde{\Phi}) = 0.$$

Stuart in [S]) has proved that for sufficiently small $\tau - \tau_{\text{cr}}$ and $|\lambda - 1|$ there exists a periodic solution of adiabatic equations, governed by H_{ad}, with

$$\{\text{zeros of } \Phi\} = \pm\sqrt{\rho}$$

so that

$$\{\text{zeros of } \tilde{\Phi}(t)\} = \pm\sqrt{\rho} e^{i\omega_0 t}$$

for some ω_0. Moreover, he proved that for $\lambda = 1 - \epsilon^2$ and sufficiently small ϵ there exists a periodic solution of dynamic equations, close to the adiabatic one and having the frequency$\sim \epsilon$ and the period $T \sim 1/\epsilon$. This justifies the adiabatic principle in this particular case.

2.3 Abrikosov strings

We can also apply the adiabatic limit method to the Euclidean model, governed by the Ginzburg-Landau energy functional in $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$, which describes Abrikosov strings in $\mathbb{R}^3$. This energy functional has the form

$$E(A, \Phi) = \frac{1}{2} \int \left\{ |dA|^2 + |d_A\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2 \right\} d^3x$$

where A is a U(1)-connection on $\mathbb{R}^3$, given by a 1-form $A = \sum_{i=1}^3 A_i dx_i$ with smooth pure imaginary coefficients $A_i = A_i(x)$, and $\Phi = \Phi(x)$ is a smooth complex-valued function on $\mathbb{R}^3$. We shall suppose further on that the gauge is chosen in such a way that $A_3 = 0$.

The Euler-Lagrange equations for $E(A, \Phi)$ have the form, similar to the 2-dimensional case

$$d^* F_A = i\mathrm{Im}(\bar{\Phi} d_A \Phi) \tag{2.3.1}$$

$$d_A^* d_A \Phi = \frac{\lambda}{2}\Phi(1 - |\Phi|^2) \tag{2.3.2}$$

A path $\xi \mapsto [A(\xi), \Phi(\xi)]$ in $\mathfrak{M}_d$ is called adiabatic if it is extremal for the energy functional $E(A, \Phi)$, restricted to paths, lying in $\mathfrak{M}_d$. The gauge fixing condition has the same form, as in $(2+1)$-dimensional case:

$$\delta^*_{(A,\Phi)}(\partial_3 A, \partial_3 \Phi) = 0.$$

As in $(2+1)$-dimensional case, we can deduce from the Euler -Lagrange equations for $E(A, \Phi)$ the adiabatic condition, having the form:

$$\left(-\partial_3^2 A, -\partial_3^2 \Phi) + \frac{1-\lambda}{2}\Phi(1 - |\Phi|^2) \right) \perp T_{(A,\Phi)}\mathfrak{M}_d$$

(the only difference with the $(2+1)$-dimensional case is another sign of the last term on the left). This is equivalent (under the gauge fixing condition) to

$$\left(-\partial_3^2 A, -\partial_3^2 \Phi) + \frac{1-\lambda}{2}\Phi(1-|\Phi|^2)\right) \perp \operatorname{Ker} \mathcal{D}_{(A,\Phi)}.$$

This is a Hamiltonian equation on $T^*\mathfrak{M}_d$ with the Hamiltonian H_{ad} of the form

$$H_{\mathrm{ad}}(A,\Phi) = \frac{1}{2}\left\{ \|\partial_3 A\|_{L^2}^2 + \|\partial_3 \Phi\|_{L^2}^2 + \frac{|1-\lambda|}{4}\|1 - \||\Phi|^2\|_{L^2}^2 \right\}.$$

Using an L^2-base $\{n_\mu\}$ of solutions of the linearized vortex equation

$$\mathcal{D}_{(A,\Phi)} n_\mu = 0 \quad , \quad \mu = 1, \ldots, 2d,$$

we can rewrite the adiabatic equation in the form

$$\langle \partial_\xi^2 (A,\Phi) + \frac{\lambda - 1}{2}\Phi(1-|\Phi|^2), n_\mu \rangle = 0 \quad , \quad \mu = 1, \ldots, 2d.$$

We call it the Abrikosov equation, its solutions describe the adiabatic limits of Abrikosov strings, slightly differing from straight lines, parallel to the (x_3)-axis.

Chapter 3

Clifford algebras and spin geometry

This chapter is a digression, containing the basic notions of the Clifford algebra and spin geometry, which shall be used in the next Chapter to define the Seiberg-Witten equations. A general reference for spin geometry is [LM].

3.1 Clifford algebras and Spin groups

3.1.1 Clifford algebras

Let V be an n-dimensional Euclidian vector space with an inner product, $\{e_i\}_{i=1}^n$ be an orthonormal basis of V. Then *Clifford algebra* $\mathrm{Cl}(V)$ is defined as an $\mathbb{R}$-algebra with unit 1, generated by $1, e_1, e_2, \cdots, e_n$, which satisfies the following relations:

$$e_i^2 = -1,$$

$$e_i e_j + e_j e_i = 0 \quad (\text{for } i \neq j).$$

Note that $V \subset \mathrm{Cl}(V)$ and

$$uv + vu = -2(u, v),$$

for $u, v \in V$. As a real vector space, $\mathrm{Cl}(V)$ has dimension 2^n and a basis, consisting of $1, e_I := e_{i_1} e_{i_2} \cdots e_{i_k}$, where $I = \{i_1, i_2, \cdots i_k\} \subset \{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_k$ and $k = |I|$.

We denote by $\mathrm{Cl}_k(V)$ the subset of the order k elements, and introduce subalgebras:

$$\mathrm{Cl}_{\mathrm{ev}} := \bigoplus_{k:\mathrm{even}} \mathrm{Cl}_k(V) \quad , \quad \mathrm{Cl}_{\mathrm{od}} := \bigoplus_{k:\mathrm{odd}} \mathrm{Cl}_k(V).$$

Then

$$\mathrm{Cl}(V) = \mathrm{Cl}_{\mathrm{ev}}(V) \oplus \mathrm{Cl}_{\mathrm{od}}(V),$$

and it provides $\mathrm{Cl}(V)$ with the structure of super-algebra.

The Clifford algebra $\mathrm{Cl}(V)$ can be provided with an inner product, extended from V, and a conjugation, defined by

$$x = \sum_{|I|=k} x_I e_I \mapsto x^* = \sum_{|I|=k} \epsilon_I x_I e_I,$$

where $\epsilon_I = (-1)^{k(k+1)/2}$ on elements of order k.

3.1.2 Universal property

The definition of the Clifford algebra $\mathrm{Cl}(V)$ does not depend on the choice of the orthonormal basis because of the following universal property, which may be taken as a definition of Clifford algebra. Namely, $\mathrm{Cl}(V)$ is a unique $\mathbb{R}$-algebra with 1 and a conjugation, which contains V, and has the following property: for any $\mathbb{R}$-algebra A with 1_A and a conjugation $a \mapsto a^*$ and for any linear map $f : V \to A$, satisfying the condition:

$$f^*(v) + f(v) = 0 \quad , \quad f^*(v)f(v) = |v|^2 1_A,$$

there exists a unique extension of f to an algebra homomorphism $f : \mathrm{Cl}(V) \to A$, preserving the conjugation.

Example 3.1.1. Here are standard examples of Clifford algebras.

1. $\mathrm{Cl}(\mathbb{R}) = \mathbb{C}$ with $e_1 = i$.

2. $\mathrm{Cl}(\mathbb{R}^2) = \mathbf{H}$ with $e_1 = i$, $e_2 = j$, $e_1 e_2 = k$.

3. $\mathrm{Cl}(\mathbb{R}^4) = \mathbf{H}[2 \times 2]$ $(2 \times 2$ matrices). What is a natural basis for this algebra?

3.1.3 Multiplicative group

Let $\mathrm{Cl}^*(V)$ be the group of invertible elements of $\mathrm{Cl}(V)$. Then $V \setminus \{0\}$ is contained in $\mathrm{Cl}^*(V)$, because $v^{-1} := -v/|v|^2$ for $v \in V^*$. The group $\mathrm{Cl}^*(V)$ acts on $\mathrm{Cl}(V)$ by the adjoint representation

$$g \mapsto \mathrm{Ad}_g(x) := gxg^{-1},$$

where $g \in \mathrm{Cl}^*(V)$. For any $u \in V \setminus \{0\}$, $v \in V$,

$$-\mathrm{Ad}_u(v) = v - \frac{2(u,v)}{|v|^2}u$$

is the reflection with respect to the hyperplane $u^\perp$.

In order to get rid of the minus sign on the left hand side of the latter formula, we introduce another action of $\mathrm{Cl}^*(V)$ on $\mathrm{Cl}(V)$, given by the twisted adjoint representation

$$g \mapsto \pi_g(x) := \alpha(g)xg^{-1},$$

where $g \in \mathrm{Cl}^*(V)$, $x \in \mathrm{Cl}(V)$ and $\alpha(g) := (-1)^{\deg g}g$ is the grading map. Then for $u \in V \setminus \{0\}$ the map $\pi_u : V \to V$ is the reflection with respect to $u^\perp$. Moreover, for $u \in V$ with $|u| = 1$,

$$\pi_u(V) = uVu^*.$$

3.1.4 Pin group

$\mathrm{Pin}(V)$ is defined as the subgroup of $\mathrm{Cl}^*(V)$, generated by unit vectors $v \in V$, i.e. by vectors v with $|v| = 1$. Since any such v generates the reflection π_v, i.e. an orthogonal transformation of V, we have a homomorphism

$$\pi : \mathrm{Pin}(V) \to \mathrm{O}(V).$$

Since any orthogonal transformation is the composition of reflections, it is an epimorphism. So we have an exact sequence:

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow \mathrm{Pin}(V) \xrightarrow{\pi} \mathrm{O}(V) \longrightarrow 0.$$

3.1.5 Spin group

$\mathrm{Spin}(V)$ is defined as the identity component of $\mathrm{Pin}(V)$, in other words,

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}_{\mathrm{ev}}(V).$$

Then we have an exact sequence:

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow \mathrm{Spin}(V) \xrightarrow{\pi} \mathrm{SO}(V) \longrightarrow 0.$$

Note that this definition of $\mathrm{Spin}(V)$ is equivalent to the following:

$$\mathrm{Spin}(V) := \{x \in \mathrm{Cl}_{\mathrm{ev}}(V) : \ x^* x = 1, \ xVx^* = V\}.$$

Example 3.1.2. Here are examples of the Spin groups.

1. $\mathrm{Spin}(\mathbb{R}) = 1$.
2. $\mathrm{Spin}(\mathbb{R}^2) = \mathrm{U}(1)$.
3. $\mathrm{Spin}(\mathbb{R}^4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$.

Exercise 3.1.3. Prove that the Lie algebra $\mathfrak{spin}(V) = \mathfrak{so}(V)$ coincides with the Lie algebra $\mathfrak{cl}_2(V)$, which is $\mathrm{Cl}_2(V)$ with the Lie bracket

$$[x, y] := xy - yx.$$

Exercise 3.1.4. Prove that for $\dim V \geq 3$ there are no non-trivial homomorphisms $\mathrm{Spin}(V) \to \mathrm{U}(1)$.

3.1.6 Spin^c groups

Let $\mathrm{Cl}^c(V) := \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified Clifford algebra, provided with a Hermitian inner product and a conjugation, extending these of $\mathrm{Cl}(V)$. We define Spin^c group as

$$\mathrm{Spin}^c(V) := \{z \in \mathrm{Cl}^c_{\mathrm{ev}}(V) : \ z^* z = 1, zVz^* = V\}.$$

We have a map

$$\pi : \mathrm{Spin}^c(V) \to \mathrm{SO}(V),$$

given by $\pi_z(v) = zvz^*$ for $v \in V$, and the following exact sequence:

$$0 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\pi} \mathrm{SO}(V) \longrightarrow 0.$$

Note that $\mathrm{Spin}^c(V)$ is a circle extension of $\mathrm{Spin}(V)$, i.e.

$$\mathrm{Spin}^c(V) = \{z = e^{i\theta} x : \ x \in \mathrm{Spin}(V), \theta \in \mathbb{R}\}.$$

So there is an exact sequence

$$0 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\delta} \mathrm{U}(1) \longrightarrow 0,$$

with $\delta : xe^{i\theta} \mapsto e^{2i\theta}$. Thus

$$\mathrm{Spin}^c(V) = \mathrm{Spin}(V) \times_{\mathbb{Z}_2} \mathrm{U}(1).$$

By the combination of the above two exact sequences, we obtain

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{(\pi,\delta)} \mathrm{SO}(V) \times \mathrm{U}(1) \longrightarrow 0\cdot$$

Note that the Lie algebra of $\mathrm{Spin}^c(V)$ is

$$\mathfrak{spin}^c = \mathfrak{cl}_2 \oplus i\mathbb{R}.$$

Example 3.1.5. We give some examples of Spin^c groups.

1. $\mathrm{Cl}^c(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C}$ and $\mathrm{Spin}^c(\mathbb{R}) = \mathrm{U}(1)$ which is embedded in $\mathbb{C} \oplus \mathbb{C}$ by the diagonal map.
2. $\mathrm{Cl}^c(\mathbb{R}^2) = \mathbb{C}[2 \times 2]$ and $\mathrm{Spin}^c(\mathbb{R}^2) = \mathrm{U}(1) \times \mathrm{U}(1)$, i.e. consists of unitary diagonal matrices in $\mathbb{C}[2 \times 2]$.

3.1.7 Spin representation

A *spin representation* is defined as a linear map

$$\Gamma : V \to \mathrm{End}\, W,$$

where V is a $2n$-dimensional Euclidian vector space and W is a 2^n-dimensional Hermitian complex vector space, which satisfies the condition:

$$\Gamma^*(v) + \Gamma(v) = 0 \quad , \quad \Gamma^*(v)\Gamma(v) = |v|^2 \mathrm{id}.$$

By the universal property, it extends to an algebra isomorphism

$$\Gamma : \mathrm{Cl}^c(V) \to \mathrm{End}\, W.$$

The action of $\mathrm{Cl}^c(V)$ on W is called the *Clifford multiplication* and elements of W are called *spinors*.

We define a *Clifford volume element* ω by

$$\omega := e_1 e_2 \cdots e_{2n} \in \mathrm{Cl}_{2n}(V).$$

Then

$$\omega^2 = (-1)^{2n} \quad , \quad \omega v + v\omega = 0 \quad \text{for all } v \in V.$$

So we can introduce the *semi-spinor spaces*

$$W^{\pm} := \{w \in W : \ \Gamma(w)w = \pm i^n w\}.$$

Then we obtain

$$W = W^+ \oplus W^-$$

and

$$\Gamma(v) : W^{\pm} \to W^{\mp} \quad \text{for all } \ v \in V.$$

Note that $W^{\pm}$ are invariant under the Clifford multiplications by even order elements.

Example 3.1.6. We give examples of spin representations.

1. $\Gamma : \mathrm{Cl}^c(\mathbb{R}^2) \to \mathbb{C}[2 \times 2]$ is the complexified Pauli map γ^c, where

$$\gamma : \mathbf{H} \ni x = (x_0, x_1, x_2, x_3) \longmapsto \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in \mathbb{C}[2 \times 2].$$

2. $\Gamma : \mathrm{Cl}^c(\mathbb{R}^4) = \mathrm{Cl}^c(\mathbf{H}) \to \mathbb{C}[4 \times 4]$ is generated by the complexified Dirac map Γ^c, where

$$\Gamma : \mathbf{H} \ni x \longmapsto \begin{pmatrix} 0 & \gamma(x) \\ -\gamma^*(x) & 0 \end{pmatrix},$$

   and γ is Pauli map. Under this map, $\mathrm{Spin}^c(\mathbf{H}) = \mathrm{Spin}^c(\mathbb{R}^4)$ is realized as

$$\begin{aligned} \mathrm{Spin}^c(\mathbb{R}^4) &= \{(U, V) \in \mathrm{U}(W^+) \times \mathrm{U}(W^-) : \det U = \det V\} \\ &= \{(U, V) \in \mathrm{U}(2) \times \mathrm{U}(2) : \ \det U = \det V\}. \end{aligned}$$

   which implies that

$$\mathrm{Spin}^c(\mathbb{R}^4) = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2 = \mathrm{Spin}(\mathbb{R}^4) \times_{\mathbb{Z}_2} \mathrm{U}(1).$$

3.1.8 Exterior algebra

Let $\Lambda^* V$ be the exterior algebra of V. We consider a map

$$\mathrm{Alt}_k : V \times \cdots \times V \to \mathrm{Cl}_k(V),$$

defined by

$$(v_1, \cdots, v_k) \mapsto \mathrm{Alt}_k(v_1, \cdots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)} \, .$$

Then we have a linear isomorphism

$$\mathrm{Alt} : \Lambda^* V \xrightarrow{\cong} \mathrm{Cl}(V)\,.$$

By duality, we also have

$$\mathrm{Alt}^* : \Lambda^*(V^*) \to \mathrm{Cl}(V^*) \cong \mathrm{Cl}(V)\,.$$

Using the spin representation $\Gamma : \mathrm{Cl}(V) \to \mathrm{End}\, W$, we can define

$$\rho := \Gamma \circ \mathrm{Alt}^* : \Lambda^*(V^*) \to \mathrm{End}\, W\,.$$

Then ρ defines the Clifford multiplication on W by forms from $\Lambda^* V^*$. In particular, the Clifford multiplication by a 2-form leaves $W_\pm$ invariant, and ρ maps real valued 2-forms to skew-Hermitian traceless endomorphisms of $W^\pm$, and imaginary 2-forms to Hermitian traceless endomorphisms of $W^\pm$.

If $\dim V = 4$, then $\Lambda^2(V^*) = \Lambda^2_+ \oplus \Lambda^2_-$ with respect to the $*$-operator and $\rho^\pm$ induces the isomorphisms:

$$\Lambda^2_\pm \xrightarrow{\cong} \mathfrak{su}(W^\pm)$$

and

$$\Lambda^2_\pm \otimes i\mathbb{R} \xrightarrow{\cong} \mathrm{Herm}\,(W^\pm).$$

We write $\sigma_\pm : \mathrm{Herm}_0(W^\pm) \to \Lambda^2_\pm \otimes \mathbb{R}$ for $(\rho_\pm)^{-1}$.

3.1.9 Kähler vector spaces

Let V be an n-dimensional complex vector space with a Hermitian metric. Then there is a canonical spin representation $(W_{\mathrm{can}}, \Gamma_{\mathrm{can}})$ with

$$W_{\mathrm{can}} = \Lambda^{0,*}(V^*) := \bigoplus_{q=0}^{n} \Lambda^{0,q}(V^*).$$

Note that in this case $V^*_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$. So for a given $v \in V$ we have the following representation for the dual covector $v^* \in V^*$

$$v^* = v^{0,1} + v^{1,0}.$$

With this notation, we define a canonical spin representation:

$$\Gamma_{\mathrm{can}} : V \to \mathrm{End}\, W_{\mathrm{can}}$$

by

$$\Gamma_{\mathrm{can}}(v) w^{0,q} := \sqrt{2}(v^{1,0} \lrcorner w^{0,q} + v^{0,1} \wedge w^{0,q})$$

for $v \in V$ and $w^{0,q} \in \Lambda^{0,q}(V^*)$. Therefore, we have

$$W^+_{\mathrm{can}} = \Lambda^{0,\mathrm{ev}}(V^*) \quad , \quad W^-_{\mathrm{can}} = \Lambda^{0,\mathrm{od}}(V^*).$$

3.2 Spin^c-structures

3.2.1 Spin^c-structure on a principal bundle

Let X be an oriented n-dimensional Riemannian manifold and $P_{\mathrm{SO}(n)} \to X$ a principal $\mathrm{SO}(n)$-bundle of orthonormal frames on X. A Spin^c-*structure* on $P_{\mathrm{SO}(n)}$ is defined as its extension to a principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c(n)} \to X$ together with a Spin^c-invariant bundle epimorphism:

$$\begin{array}{ccc} P_{\mathrm{Spin}^c(n)} & \longrightarrow & P_{\mathrm{SO}(n)} \\ \downarrow & & \downarrow \\ X & = & X, \end{array}$$

where $\mathrm{Spin}^c(n)$ acts on $P_{\mathrm{SO}(n)}$ by

$$\pi : \mathrm{Spin}^c(n) \to \mathrm{SO}(n).$$

We can define an associated principal U(1)-bundle $P_{\mathrm{U}(1)} \to X$ such that

$$\begin{array}{ccc} P_{\mathrm{Spin}^c(n)} & \xrightarrow{\delta} & P_{\mathrm{U}(1)} \\ \downarrow & & \downarrow \\ X & = & X, \end{array}$$

where $\mathrm{Spin}^c(n)$ acts on $P_{\mathrm{U}(1)}$ by

$$\delta : \mathrm{Spin}^c(n) \to \mathrm{U}(1).$$

The complex line bundle $L \to X$, associated with $P_{\mathrm{U}(1)} \to X$, is called the *characteristic bundle* of the Spin^c-structure, and its 1st Chern class $c_1(L)$ is the *characteristic class* of the Spin^c-structure.

3.2.2 Spin^c-structure on a vector bundle

In analogous way, one can define a Spin^c-structure on an oriented Riemannian vector bundle $V \to X$ of rank n, associated with $P_{\mathrm{SO}(n)} \to X$, that is, isomorphic to $V \cong P_{\mathrm{SO}(n)} \times_{\mathrm{SO}(n)} \mathbb{R}^n$. A Spin^c-structure on $V \to X$ is an extension of its structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}^c(n)$. In other words, $V \to X$ admits a Spin^c-structure if it is associated with the principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c(n)} \to X$, i.e. there exists a bundle isomorphism

$$P_{\mathrm{Spin}^c(n)} \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n \longrightarrow V$$

where $\mathrm{Spin}^c(n)$ acts on $\mathbb{R}^n$ by the homomorphism $\pi : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$.

In particular, one can take for V the tangent bundle TX. In this case, a Spin^c-structure on TX is called a Spin^c-structure on X.

When rank $V = 2n$, we can give an equivalent definition of a Spin^c-structure on V in terms of the spin representation. Namely, using this representation, we can construct in this case a complex 2^n-rank Hermitian vector bundle W, associated with the principal $\mathrm{Spin}^c(2n)$-bundle $P_{\mathrm{Spin}^c(2n)} \to X$:

$$W := P_{\mathrm{Spin}^c(2n)} \times_{\mathrm{Spin}^c(2n)} \mathbb{C}^{2^n} \longrightarrow X$$

where the action of $\mathrm{Spin}^c(2n)$ on $\mathbb{C}^{2^n}$ is given by the spin representation

$$\Gamma : \mathrm{Spin}^c(2n) \longrightarrow \mathrm{End}\,\mathbb{C}^{2^n}.$$

This representation yields a linear bundle homomorphism (denoted by the same letter)

$$\Gamma : V \to \mathrm{End}\,W$$

which satisfies the characteristic properties of spin representations above. We call W the *spinor bundle.*

So the definition of Spin^c-structure on V in this case is equivalent to the following. A Spin^c-structure on V of rank $2n$ is a pair (W, Γ), consisting of a complex Hermitian vector bundle $W \to X$ of rank 2^n and a bundle homomorphism $\Gamma : V \to \mathrm{End}\,W$, having the spin representation properties

$$\Gamma^*(v) + \Gamma(v) = 0 \quad , \quad \Gamma^*(v)\Gamma(v) = |v|^2\mathrm{id}.$$

The bundle homomorphism $\Gamma : V \to \mathrm{End}\,W$ can be extended to a bundle homomorphism

$$\Gamma : \mathrm{Cl}^c(V) \longrightarrow \mathrm{End}\,W$$

where $\mathrm{Cl}^c(V)$ is the complexified Clifford algebra bundle, associated with the oriented Riemannian vector bundle V. Then W can be decomposed into the direct sum of $\Gamma(\omega)$-eigenbundles

$$W = W^+ \oplus W^-,$$

called the *semi-spinor bundles.* The characteristic line bundle of a Spin^c-structure (W, Γ) can be defined as

$$L_\Gamma := P_{\mathrm{Spin}^c(2n)} \times_{\mathrm{Spin}^c(2n)} \mathbb{C} \longrightarrow X$$

where the action of $\mathrm{Spin}^c(2n)$ on $\mathbb{C}$ is given by the homomorphism

$$\delta : \mathrm{Spin}^c(2n) \longrightarrow \mathrm{U}(1).$$

Exercise 3.2.1. Prove that $L_\Gamma^{\otimes 2^{n-2}}$ is isomorphic to the determinant line bundles $\det W^+ \cong \det W^-$. In particular, for $n = 2$: $L_\Gamma \cong \det W^+ \cong \det W^-$.

3.2.3 The existence of Spin^c-structures and the space of Spin^c structures

It can be proved that $P_{\mathrm{SO}(n)}$ *admits a Spin^c-structure if and only if there exists a* $c \in H^2(X, \mathbb{Z})$ *such that*

$$w_2(P_{\mathrm{SO}(n)}) \equiv c \pmod 2,$$

where w_2 is the second Stiefel-Whitney class. This is proved by using the exact sequence:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1) \longrightarrow 0.$$

It is worthwhile to compare this criterion with the necessary and sufficient condition of existence of a Spin-structure on the principal bundle $P_{\mathrm{SO}(n)} \to X$. The latter condition is

$$w_2(P_{\mathrm{SO}(n)}) = 0.$$

It follows, in particular, that a Spin^c-structure exists on any spin manifold and any almost complex manifold X (in the latter case take for $c = c_1(X)$).

Exercise 3.2.2. Prove that a Spin^c-structure exists on any oriented compact four-manifold X, using the fact that for such manifolds,

$$w_2(X) \cdot \alpha \equiv \alpha \cdot \alpha \pmod 2$$

for all $\alpha \in H_2(X, \mathbb{Z})$.

Suppose that an oriented Riemannian vector bundle $V \to X$ of rank $2n$ has a Spin^c-structure (W, Γ). Then for any complex line bundle $E \to X$, we can define a new Spin^c-structure (W_E, Γ_E) by setting

$$W_E := W \otimes E \quad , \quad \Gamma_E := \Gamma \otimes \mathrm{id}.$$

Then the new Spin^c-structure (W_E, Γ_E) will correspond to the principal Spin$^c(2n)$-bundle

$$P_{\Gamma_E} = P_\Gamma \otimes_{\mathrm{U}(1)} P_E$$

where P_Γ is the principal Spin$^c(2n)$-bundle, associated with (W, Γ), and P_E is the principal U(1)-bundle, associated with E. The characteristic bundle of (W_E, Γ_E) is equal to

$$L_{\Gamma_E} := L_\Gamma \otimes E^{\otimes 2}.$$

Exercise 3.2.3. Prove that any Spin^c-structure on $V \to X$ can be obtained in this way.

Thus the space of all Spin^c-structures on V, if it's not empty, is parametrized by $H^2(X, \mathbb{Z})$ (though not canonically). However, in the almost complex case, we can define a canonical Spin^c-structure $(W_{\text{can}}, \Gamma_{\text{can}})$ and make this identification canonical.

3.2.4 Spin^c-structures on almost complex vector bundles

Suppose that $V \to X$ is an almost complex vector bundle of (complex) rank n, provided with an almost complex structure J, compatible with the Riemannian metric and orientation of V. Then V has a canonical Spin^c-structure $(W_{\text{can}}, \Gamma_{\text{can}})$, which can be defined by setting

$$W_{\text{can}} := \Lambda^{0,*} V^*$$

where V^* is provided with the dual almost complex structure J^*. The Clifford multiplication map Γ_{can} is given by the same formula, as in the case of Kähler vector spaces. The characteristic bundle L_{can} coincides with the anticanonical bundle K^* of V:

$$K^* = \Lambda^{0,n}(V^*).$$

Any other Spin^c-structure on V is obtained from the canonical one by multiplying it by a Hermitian line bundle $E \to X$ so that

$$W_E = W_{\text{can}} \otimes E \quad , \quad K^* \otimes E^2,$$

so the space of Spin^c-structures on V is canonically identified with $H^2(X, \mathbb{Z})$.

3.3 Spin^c-connections and Dirac operators

3.3.1 Spin^c-connections in terms of principal bundles

Let X be an oriented Riemannian manifold of dimension $2n$, provided with a Spin^c-structure (W, Γ). We denote by ∇ the Levi-Civita connection on TX, generated by the Riemannian metric on X. Then a Spin^c-*connection* is an extension of ∇ to W. More precisely, it is a connection ∇ on W, satisfying the following relation:

$$\nabla_u(\Gamma(v)\Phi) = \Gamma(v)\nabla_u\Phi + \Gamma(\nabla_u v)\Phi$$

for any $u, v \in \mathrm{Vect}(X)$, $\Phi \in C^\infty(X, W)$. Then such a connection preserves the semi-spinor bundles $W^\pm$ and any two such connections differ by an imaginary valued 1-form on X.

In principal bundle terms, let $P_{\mathrm{SO}(2n)} \to X$ be the frame bundle of X, and $P_\Gamma := P_{\mathrm{Spin}^c(2n)} \to X$ be its extension to a principal $\mathrm{Spin}^c(2n)$-bundle over X, associated with a Spin^c-structure (W, Γ). Then we have

$$W = P_{\mathrm{Spin}^c(2n)} \times_{\mathrm{Spin}^c(2n)} W_0,$$

where $W_0 = \mathbb{C}^{2^n}$, and $\mathrm{Spin}^c(2n)$ acts on W_0 by the standard spin representation Γ_0. Also we have

$$TX = P_{\mathrm{Spin}^c(2n)} \times_{\mathrm{Spin}^c(2n)} V_0,$$

where $V_0 = \mathbb{R}^{2n} = \mathbb{C}^n$ and $\mathrm{Spin}^c(2n)$ acts on V_0 by the homomorphism $\pi : \mathrm{Spin}^c(2n) \to \mathrm{SO}(2n)$, and

$$L_\Gamma = P_{\mathrm{Spin}^c(2n)} \times_{\mathrm{Spin}^c(2n)} \mathbb{C},$$

where $\mathrm{Spin}^c(2n)$ acts on $\mathbb{C}$ by

$$\delta : \mathrm{Spin}^c(2n) \to \mathrm{U}(1).$$

Consider the standard spin representation

$$\Gamma_0 : \mathrm{Cl}^c(V_0) \to \mathrm{End}\, W_0,$$

and denote by G a subgroup in $\mathrm{Aut}\, W_0$, which is the image of $\mathrm{Spin}^c(2n)$ under Γ_0: $G = \Gamma_0(\mathrm{Spin}^c(2n))$. Its Lie algebra is

$$\mathfrak{g} := \mathrm{Lie}\, G = \Gamma_0(\mathfrak{cl}_2(V_0) \oplus i\mathbb{R}) = \Gamma_0(\mathfrak{cl}_2(V_0)) \oplus i\mathbb{R} = \mathfrak{g}_0 \oplus i\mathbb{R}$$

where $\mathfrak{g}_0 := \Gamma_0(\mathfrak{cl}_2(V_0))$.

Then a Spin^c-connection on W is generated by a connection 1-form $\mathcal{A} \in \Omega^1(P_\Gamma, \mathfrak{g})$. We can write

$$\mathcal{A} = \mathcal{A}_0 \oplus A,$$

where $\mathcal{A}_0 \in \Omega^1(P_\Gamma, \mathfrak{g}_0)$ is the traceless part of $\mathcal{A}$, and $A \in \Omega^1(P_\Gamma, i\mathbb{R})$ is the trace part, that is, $A = \mathrm{Tr}\, A/2^n$. The traceless part $\mathcal{A}_0$ generates a connection on TX since $\mathfrak{g}_0 = \mathfrak{so}(2n)$, and by definition of Spin^c-connection, it should coincide with the Levi-Civita connection. Hence, $\mathcal{A}$ is completely determined by its trace part $A \in \Omega^1(P_\Gamma, i\mathbb{R})$. Since $\delta(e^{i\theta} \cdot 1) = e^{2i\theta}$, the trace part $A \in \Omega^1(P_\Gamma, i\mathbb{R})$ generates the connection $2A$ on the characteristic bundle $L_\Gamma (= L)$. If L has a square root $L^{1/2} \to X$ (it is so if, e.g., X is a spin manifold), then A also generates a connection on $L^{1/2}$. However, in general, A can be considered only as a virtual connection on the virtual line bundle $L^{1/2}$. We denote by $\mathcal{A}(\Gamma)$ the space of such virtual connections A on the virtual line bundle $L^{1/2}$.

3.3.2 Dirac operator

We denote by ∇_A (respectively, d_A) the covariant derivative (respectively, the exterior covariant differentiation) on sections of W, generated by $\mathcal{A} = \mathcal{A}_0 + A$. Then the *Dirac operator*

$$D_A : C^\infty(X, W^+) \to C^\infty(X, W^-),$$

associated with a virtual connection A, is given by the following formula:

$$D_A\Phi = \sum_{\nu=1}^{2n} \Gamma(e_\nu)\nabla_{A,e_\nu}\Phi,$$

where $\Phi \in C^\infty(X, W^+)$, $\{e_\nu\}$ is a local orthonormal basis of TX.

Exercise 3.3.1. Why this definition does not depend on the choice of e_ν?

We can also define by duality the adjoint Dirac operator:

$$D_A^* : C^\infty(X, W^-) \to C^\infty(X, W^+).$$

3.3.3 Spin^c-connections and Dirac operator on almost complex manifolds

Let (X, J) be a $2n$-dimensional (over $\mathbb{R}$) almost complex manifold with an almost complex structure J, compatible with the orientation and Riemannian metric g. We denote by $(W_{\mathrm{can}}, \Gamma_{\mathrm{can}})$ the canonical Spin^c-structure on X.

If J is integrable and parallel with respect to g, i.e. X is Kähler, then the Levi-Civita connection $\nabla = \nabla_g$ preserves the space $\Omega^{0,q}(X)$ and it can be extended to the canonical Spin^c-connection ∇_{can} on W_{can}. In particular, $2A_{\mathrm{can}}$ is the canonical connection on the canonical bundle

$$L_{\mathrm{can}} = K^*(X) = \Lambda^{0,n}(T^*X).$$

If J is not integrable, then the Levi-Civita connection does not preserve the spaces $\Omega^{0,q}$, but one can still define a canonical Spin^c-connection ∇_{can} in W_{can}, modifying the Levi-Civita connection by adding a term, containing the Nijenhuis tensor of J.

Any other Spin^c-structure on (X, J) has the form

$$W_E = W_{\mathrm{can}} \otimes E,$$

where E is a Hermitian line bundle $E \to X$. Accordingly, any Spin^c-connection on W_E has the form

$$A = A_{\mathrm{can}} \otimes \mathrm{id} + \mathrm{id} \otimes B,$$

where B is a Hermitian connection on $E \to X$. Then the corresponding Dirac operator

$$D_A : C^\infty(X, W_E^+) \to C^\infty(X, W_E^-),$$

where $W_E^+ = \Lambda^{0,\mathrm{ev}}(X,E)$, $W_E^- = \Lambda^{0,\mathrm{od}}(X,E)$, will be equal to

$$D_A = \sqrt{2}(\bar{\partial}_B + \bar{\partial}_B^*).$$

Exercise 3.3.2. Prove this formula.

3.3.4 Weitzenböck formula

Let X be an oriented $2n$-dimensional Riemannian manifold, and

$$\nabla_A^* : C^\infty(X, W) \to C^\infty(X, W)$$

be the L^2-adjoint operator of ∇_A. Then the Weitzenböck formula reads as follows

$$D_A^* D_A \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{4} s \Phi + \rho^+(F_A)\Phi,$$

$$D_A D_A^* \Psi = \nabla_A \nabla_A^* \Psi + \frac{1}{4} s \Psi + \rho^-(F_A)\Psi,$$

where $\Phi \in C^\infty(X, W^+)$, $\Psi \in C^\infty(X, W^-)$, s is the scalar curvature of (X, g), and

$$\rho^\pm : \Lambda^2(T^*X) \otimes \mathbb{C} \to \mathrm{End}_0\,(W^\pm)$$

are the maps, introduced in Sec.3.1. (Here $\mathrm{End}_0\,(W^\pm)$ denotes the space of Hermitian traceless endomorphisms of $W^\pm$).

Chapter 4

Dimension four — Seiberg-Witten equations

In this Chapter we deal with the Seiberg-Witten equations on compact, oriented, Riemannian four-manifolds and their solutions. In Sec.4.1 we review some general properties of these equations and moduli spaces of their solutions. Sec.4.2 is devoted to the special case — Kähler surfaces, in which we can give a description of the moduli space in terms of complex curves, similar to that in the Bradlow's theorem in the case of vortex equations on compact Riemann surfaces. In the next sections we turn to the Seiberg-Witten equations on symplectic 4-manifolds. We start from their general properties in Sec.4.3 and then switch to the Taubes' correspondence between solutions of Seiberg-Witten equations and pseudoholomorphic curves. In Sec.4.4 we discuss the direct construction, associating a pseudoholomorphic curve with the scale limit of a Seiberg-Witten solution. In Sec.4.5 we consider the inverse correspondence, assigning to a section of a vortex bundle over a pseudoholomorphic curve an approximate solution of Seiberg-Witten equations. A general reference for this and next sections is [Mo],[Sal].

4.1 Seiberg-Witten equations on Riemannian 4-manifolds

4.1.1 Seiberg-Witten equations

Let X be a compact, oriented, Riemannian four-manifold. Suppose that it is provided with a Spin^c-structure (W, Γ) and a Spin^c-connection ∇_A, generated by a virtual connection $A \in \mathcal{A}(\Gamma)$ on L_Γ.

We introduce the following *Seiberg-Witten equations*:

$$D_A\Phi = 0, \tag{4.1.1}$$

$$F_a^+ = \sigma^+(\Phi\otimes\Phi^*)_0, \tag{4.1.2}$$

where

$$\sigma^+(\Phi\otimes\Phi^*)_0 := \sigma^+(\Phi\otimes\Phi^* - 1/2|\Phi|^2\,\mathrm{id}),$$

and $\Phi\otimes\Phi^* - 1/2|\Phi|^2\mathrm{id}$ is a traceless Hermitian endomorphism of W^+, associated with Φ, and

$$\sigma^+ := (\rho^+)^{-1} : \mathrm{Herm}_0(W^+) \xrightarrow{\cong} \Omega^2_+(X, i\mathbb{R}).$$

4.1.2 The Seiberg-Witten functional

We introduce the following *Seiberg-Witten energy functional*

$$E(A,\Phi) = \frac{1}{2}\int_X \left\{ |F_A|^2 + |\nabla_A\Phi|^2 + \frac{|\Phi|^2}{4}(s+|\Phi|^2)\right\} d\,\mathrm{vol}\,, \tag{4.1.3}$$

where $s := s(g)$ denotes the scalar curvature of (X,g). Note that $E(A,\Phi)$ can be negative if s is negative.

Using the Weitzenböck formula, we can prove the following Bogomol'nyi formula

$$E(A,\Phi) = \frac{1}{2}\int_X \{|D_A\Phi|^2 + 2|F_A^+ - \sigma^+(\Phi\otimes\Phi^*)_0|^2\}\,d\,\mathrm{vol} - \frac{\pi^2}{2}\langle c_1(L_\Gamma)^2, [X]\rangle.$$

To prove this formula, we use the following formula of Chern-Weil type

$$\pi^2\langle c_1(L_\Gamma)^2, [X]\rangle = -\int_X F_A\wedge F_A = \|F_A^+\|^2 - \|F_A^-\|^2.$$

On the other hand

$$|F_A^+ - \sigma^+(\Phi\otimes\Phi^*)_0|^2 = |F_A^+|^2 + |\sigma^+(\Phi\otimes\Phi^*)_0|^2 - \\ 2\langle F_A^+, \sigma^+(\Phi\otimes\Phi^*)_0\rangle = |F_A^+|^2 + \tfrac{1}{8}|\Phi|^4 - \tfrac{1}{2}\langle\rho^+(F_A)\Phi,\Phi\rangle$$

and by the Weitzenböck formula

$$\|D_A\Phi\|^2_{L^2} = \|\nabla_A\Phi\|^2_{L^2} + \frac{1}{4}\int_X s|\Phi|^2 d\mathrm{vol} + \langle\rho^+(F_A)\Phi,\Phi\rangle_{L^2}.$$

These three relations imply the Bogomol'nyi formula.

The Bogomol'nyi formula yields the following inequality

$$E(A,\Phi) \geq -\frac{\pi^2}{2}\langle c_1(L_\Gamma)^2, [X]\rangle$$

and the equality here is achieved only on solutions of Seiberg-Witten equations.

4.1.3 Gauge transformations and perturbed Seiberg-Witten equations

The Seiberg-Witten equations, as well as the Seiberg-Witten functional $E(A, \Phi)$, are invariant under gauge transformations, given by

$$A \mapsto A + u^{-1}du \quad , \quad \Phi \mapsto u^{-1}\Phi$$

where $u = e^{i\chi}$, χ is a real-valued function, so that $u \in \mathcal{G} := C^\infty(X, \mathrm{U}(1))$. This action is free, unless $\Phi \equiv 0$. In order to avoid solutions of the form $(A, 0)$, we perturb the Seiberg-Witten equations as follows.

$$D_A \Phi = 0, \tag{4.1.4}$$

$$F_A^+ + \eta = \sigma^+(\Phi \otimes \Phi^*)_0, \tag{4.1.5}$$

where $\eta \in \Omega_+^2(X, \mathbf{R})$. We call them briefly the SW_η-equations. Note that if $b_+^2 := \dim H_+^2(X, \mathbf{R}) \geq 1$, we can always find an η such that the SW_η-equations have no solutions of type $(A, 0)$. It follows from

Exercise 4.1.1. Introduce the Γ-wall by

$$\Omega_\Gamma^2(X, i\mathbf{R}) := \{\eta \in \Omega_+^2(X, i\mathbf{R}) | \exists A \in \mathcal{A}(\Gamma) \quad \text{with} \quad F_A^+ + \eta = 0\}.$$

Prove that $\Omega_\Gamma^2(X, i\mathbf{R})$ is an affine vector subspace in $\Omega_+^2(X, i\mathbf{R})$ of codimension b_2^+.

4.1.4 Moduli space of solutions

The moduli space of solutions of Seiberg-Witten equations is defined, as before:

$$\mathcal{M}_\eta(X, \gamma, g) := \{\mathrm{SW}_\eta\text{-solutions } (A, \Phi)\}/\mathcal{G}.$$

If $b_+^2 \geq 1$, then $\mathcal{M}_\eta(X, \gamma, g)$ is smooth for an appropriate η.

Theorem 4.1.2. *If $b_+^2 > 1$, then for a generic $\eta \in \Omega_+^2(X, \mathbf{R})$ the moduli space $\mathcal{M}_\eta(X, \gamma, g)$ is a compact, oriented, smooth manifold of dimension*

$$\dim \mathcal{M}_\eta(X, \Gamma, g) = \frac{\langle c_1(L_\Gamma)^2, [X] \rangle - 2\chi(X) - 3\sigma(X)}{4},$$

where $\chi(X)$ is the Euler characteristic of X and $\sigma(X)$ is the signature of $H^2(X)$.

According to this theorem, the homology class $[\mathcal{M}_\eta(X,\Gamma,g)]$ in the (infinite dimensional) configuration space $\{(A,\Phi)\}/\mathcal{G}$ is correctly defined and does not depend on the choice of generic η and g. It depends only on the Spin^c-structure Γ. (In the boundary case $b_+^2 = 1$ the moduli space $\mathcal{M}_\eta$ does depend on η since the Γ-wall is of codimension 1 and divides the space $\Omega_+^2(X, i\mathbb{R})$ into two connected components).

Suppose, in particular, that $\dim \mathcal{M}_\eta(X,\Gamma,g) = 0$, that is,

$$\langle c_1(L_\Gamma)^2, [X]\rangle = 2\chi(X) + 3\sigma(X)$$

(this condition arises also in the Wu's theorem on the existence of an almost complex structure on a given Riemannian manifold). Then the moduli space $\mathcal{M}_\eta(X,\Gamma,g)$ consists of a finite union of points with a sign. In this case we define the *Seiberg-Witten invariant* $\mathrm{SW}(X,\Gamma)$ by

$$\mathrm{SW}(X,\Gamma) := \sum_{\text{points} \in \mathcal{M}_\eta} \text{signs} \quad \in \mathbf{Z}.$$

It is invariant under orientation-preserving diffeomorphisms f of X in the sense that

$$\mathrm{SW}(X,\Gamma) = \mathrm{SW}(f(X), f^*\Gamma).$$

4.1.5 Scale transformations

The SW_η-equations are not invariant under the change: $g \mapsto \lambda^2 g$ of scale of the underlying Riemannian metric. More precisely, there is a one-to-one correspondence between

$$\{\mathrm{SW}_\eta\text{-solutions } (A,\Phi) \text{ for metric } g\}$$

and

$$\{\mathrm{SW}_\eta\text{-solutions } (A,\frac{1}{\lambda}\Phi) \text{ for metric } \lambda^2 g\},$$

where $\lambda > 0$ is a constant. Note that the Seiberg-Witten functional under the scale change transforms as

$$E_g(A,\Phi) = E_{\lambda^2 g}(A, \frac{1}{\lambda}\Phi).$$

4.2 Seiberg-Witten equations on Kähler surfaces

4.2.1 Seiberg-Witten equations

Let (X, ω, J) be a compact Kähler surface, provided with the canonical Spin^c-structure $(W_{\mathrm{can}}, \Gamma_{\mathrm{can}})$ and canonical Spin^c-connection $\nabla_{\mathrm{can}} = \nabla_{A_{\mathrm{can}}}$, where $2A_{\mathrm{can}}$ is a connection on the anticanonical bundle K^*.

Then any other Spin^c-structure on X is associated with some Hermitian line bundle $E \to X$ so that the semi-spinor bundles are given by

$$W_E^+ = W_{\mathrm{can}}^+ \otimes E = \Lambda^0(E) \oplus \Lambda^{0,2}(E) \quad , \quad W_E^- = W_{\mathrm{can}}^- \otimes E = \Lambda^{0,1}(E).$$

The characteristic bundle coincides with

$$L_{\Gamma_E} = L_{\mathrm{can}} \otimes E^2 = K^* \otimes E^2.$$

A Spin^c-connection ∇_A on W_E can be written as $\nabla_A = \nabla_{\mathrm{can}} + B$, where B is a Hermitian connection on $E \to X$. Furthermore, in this case, the Dirac operator can be written as

$$D_A = \sqrt{2}(\bar{\partial}_B + \bar{\partial}_B^*)$$

for $\Phi = (\varphi_0, \varphi_2) \in \Omega^0(X, E) \oplus \Omega^{0,2}(X, E)$.

The right-hand-side of the Seiberg-Witten curvature equation (4.1.2) is rewritten as follows:

$$\sigma^+(\Phi \otimes \Phi^*) = i\frac{|\varphi_0|^2 - |\varphi_2|^2}{4}\omega + \frac{\bar{\varphi}_0\varphi_2 - \varphi_0\bar{\varphi}_2}{2}.$$

Recall that for Kähler surfaces we have the decomposition

$$\Lambda_+^2 \otimes \mathbb{C} = \Lambda^{2,0} \oplus \mathbb{C}[\omega] \oplus \Lambda^{0,2}.$$

Accordingly, the Seiberg-Witten curvature equation decomposes into the component, parallel to ω, the (0,2)-component and the (2,0)- component, which is conjugate to the (0,2)-component.

Hence the SW$_\eta$-equations on a compact Kähler surface are rewritten as follows:

$$\bar{\partial}_B\varphi_0 + \bar{\partial}_B^*\varphi_2 = 0, \tag{4.2.1}$$

$$F_B^{0,2} + \eta^{0,2} = \frac{\bar{\varphi}_0\varphi_2}{2}, \tag{4.2.2}$$

$$F^{\omega}_{A_{\mathrm{can}}} + F^{\omega}_B = \frac{i}{4}(|\varphi_0|^2 - |\varphi_2|^2) - \eta^{\omega}. \tag{4.2.3}$$

The first of these equations is the Dirac equation, the second one is the (0,2)-component of the curvature equation and the third one is the component of the curvature equation, parallel to ω.

4.2.2 Solvability conditions

Hereafter, we will assume that η is of the (1,1)-type and apply $\bar{\partial}_B$-operator to the 1st Seiberg-Witten equation

$$\begin{aligned} \bar{\partial}_B \bar{\partial}^*_B \varphi_2 &= -\bar{\partial}_B \bar{\partial}_B \varphi_0 \quad (\text{ use } (4.2.1)) \\ &= -F^{0,2}_B \varphi_0 \quad (\text{definition of } F^{0,2}_B) \\ &= -\frac{|\varphi_0|^2 \varphi_2}{2} \quad (\text{use } (4.2.2)). \end{aligned} \tag{4.2.4}$$

Taking the Hermitian inner product of (4.2.4) with φ_2 and integrating it over X, we obtain

$$||\bar{\partial}^*_B \varphi_2||_{L^2} + \frac{||\varphi_0||^2_{L^2} ||\varphi_2||^2_{L^2}}{2} = 0. \tag{4.2.5}$$

Thus

$$\bar{\partial}^*_B \varphi_2 = \bar{\partial}_B \varphi_0 = \bar{\varphi}_0 \varphi_2 \equiv 0.$$

Hence, either φ_0 or φ_2 should be identically zero. In order to decide which of these is identically zero, we integrate the third equation. Then

$$\begin{aligned} \int_X \frac{|\varphi_0|^2 - |\varphi_2|^2}{4} \omega \wedge \omega &= i \int_X (F_{A_{\mathrm{can}}} + F_B + \eta) \wedge \omega \\ &= \pi(-c_1(K) + 2c_1(E)) \cdot [\omega] + i \int_X \eta \wedge \omega. \end{aligned} \tag{4.2.6}$$

Note that

$$\int_X \frac{|\varphi_0|^2 - |\varphi_2|^2}{4} \omega \wedge \omega = \frac{||\varphi_0||^2_{L^2} - ||\varphi_2||^2_{L^2}}{2}.$$

Consider, in particular, the case $\eta = 0$, i.e. the non-perturbed Seiberg-Witten equations. Then

$$||\varphi_2||^2 - ||\varphi_0||^2 = 2\pi(2c_1(E) \cdot [\omega] - c_1(K) \cdot [\omega])$$

and we obtain the following solvability conditions:

- if $c_1(E) \cdot [\omega] > c_1(K) \cdot [\omega]/2$, then $\varphi_0 \equiv 0$, $\varphi_2 \not\equiv 0$.

- if $c_1(E) \cdot [\omega] < c_1(K) \cdot [\omega]/2$, then $\varphi_0 \not\equiv 0$, $\varphi_2 \equiv 0$.

Note that for a Kähler surface with $b_2^+ > 1$ we have the inequality

$$c_1(K) \cdot [\omega] \geq 0,$$

since the canonical bundle K of such a surface has a non-trivial holomorphic section. By the same reason,

- if $(E, \bar{\partial}_B)$ has a non-trivial holomorphic section φ_0, then $c_1(E) \cdot [\omega] \geq 0$

- if $K \otimes E^*$ has a non-trivial holomorphic section φ_2, then $c_1(K) \cdot [\omega] \geq c_1(E) \cdot [\omega]$.

Figure 4.1: the solvability diagram for $\eta = 0$

4.2.3 The case of trivial E

Next we consider the SW_η-equations for the trivial E and take

$$\eta = -F^+_{A_{\mathrm{can}}} + i\lambda\omega,$$

where $\lambda > 0$. From (4.2.3), we obtain

$$4i(dB)^\omega = 4\lambda + |\varphi_2|^2 - |\varphi_0|^2.$$

Integrating this over X, we get

$$4\lambda \mathrm{Vol}\,(X) + ||\varphi_2||^2 - ||\varphi_0||^2 = 0.$$

Thus $\varphi_2 \equiv 0$ and the SW_η-equations have the form

$$\bar\partial_B \varphi_0 = 0, \tag{4.2.7}$$

$$F_B^{0,2} = 0, \tag{4.2.8}$$

$$4i(dB)^\omega = 4\lambda - |\varphi_0|^2. \tag{4.2.9}$$

Since E is trivial, these equations have a trivial solution:

$$B \equiv 0,\ \varphi_0 \equiv 2\sqrt{\lambda},\ \varphi_2 \equiv 0.$$

Using the fact that these equations are of Liouville type (cf. 4.2.7, 4.2.8, 4.2.9), it may be proved that this solution is unique (up to gauge). It follows that in this case

$$\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1.$$

4.2.4 Description of the moduli space in terms of effective divisors

We show that for the SW_η-equations on a Kähler surface there exists an analogue of Bradlow's theorem for vortex equations on compact Riemann surfaces. Let $E \to X$ be a Hermitian line bundle over (X, ω, J). Suppose that for some $\lambda > 0$ its 1st Chern class satisfies the inequality

$$0 \leq c_1(E) \cdot [\omega] < \frac{c_1(K) \cdot [\omega]}{2} + \lambda \mathrm{Vol}\,(X). \tag{4.2.10}$$

This inequality plays the same role as the stability condition $c_1(L) < \tau/4\pi \operatorname{Vol}_g(X)$ in Bradlow's theorem 1.4.1.

Under this condition, the moduli space of SW_η-solutions for $\eta = \pi i\lambda\omega$ and Spin^c-structure (W_E, Γ_E) admits the following description: *there is a one-to-one correspondence between the gauge equivalent classes of SW_η-solutions (B, φ_0) and effective divisors of* deg $= c_1(E)$ *on* X. *The latter space can be identified with the space of complex gauge equivalent classes of holomorphic line bundles $(E, \bar\partial_E)$ with a non-trivial holomorphic section φ_0. Since $\bar\partial_E = \bar\partial_B$ for some Hermitian connection B, this space coincides with the space of solutions (B, φ_0) of the equations*

$$\bar\partial_B \varphi_0 = 0 \quad , \quad F_B^{0,2} = 0$$

modulo complex gauge transformations (cf. 4.2.1, 4.2.2).

To prove this equivalence, we should prove that for any solution (B, φ) of the above equations there exists a unique $\mathcal{G}_{\mathbb{C}}$-equivalent solution (B_u, φ_u), satisfying the third Seiberg-Witten equation (4.2.3). Writing down the gauge factor u in the form $u = e^\theta$ for a real-valued function $\theta \in \mathbb{R}$, we obtain the following Liouville-type equation for θ:

$$8i(\partial\bar\partial\theta)^\omega + e^{-2\theta}|\varphi_0|^2 = 4\pi\lambda - 4i(F_B^\omega + F_{A_c})^\omega.$$

According to the Kazdan-Warner theorem, this equation has a unique solution under our condition (4.2.10).

4.3 Seiberg-Witten equations on symplectic four-manifolds

4.3.1 Seiberg-Witten equations

Let (X, ω, J) be a compact symplectic four-manifold together with a compatible almost complex structure J. Let (W_E, Γ_E) be a Spin^c-structure on X, associated with a Hermitian line bundle $E \to X$, which is provided with a Hermitian connection B.

The corresponding SW_η-equations have the form

$$\bar\partial_B \varphi_0 + \bar\partial_B^* \varphi_2 = 0, \tag{4.3.1}$$

$$F_{A_{\mathrm{can}}}^{0,2} + F_B^{0,2} + \eta^{0,2} = \frac{\bar\varphi_0 \varphi_2}{2}, \tag{4.3.2}$$

$$F^{\omega}_{A_{\mathrm{can}}} + F^{\omega}_{B} + \eta^{\omega} = \frac{|\varphi_2|^2 - |\varphi_0|^2}{4} \tag{4.3.3}$$

where $(\varphi_0, \varphi_2) \in \Omega^0(X, E) \oplus \Omega^{0,2}(X, E)$. Note that $F_{A_{\mathrm{can}}}$ is not necessarily of type $(1, 1)$ for a general almost complex structure J.

Consider again, as in the Kähler case, the perturbation η of the form

$$\eta = -F^{+}_{A_{\mathrm{can}}} + \pi i \lambda \omega,$$

where λ is a positive number, and introduce the normalized sections:

$$\alpha := \frac{\varphi_0}{\sqrt{\lambda}} \quad , \quad \beta := \frac{\varphi_2}{\sqrt{\lambda}}.$$

Then the SW$_{\eta}$-equations become

$$\bar{\partial}_B \alpha + \bar{\partial}^*_B \beta = 0, \tag{4.3.4}$$

$$\frac{2}{\lambda} F^{0,2}_B = \bar{\alpha}\beta, \tag{4.3.5}$$

$$\frac{4i}{\lambda} F^{\omega}_B = 4\pi + |\beta|^2 - |\alpha|^2. \tag{4.3.6}$$

We call them briefly the *SW_{λ}-equations.*

4.3.2 Solvability conditions

Now we examine the solvability conditions for the SW$_{\lambda}$-equations. They look the same, as in the Kähler case, but, in contrast with the latter case, we have now

$$\bar{\partial}_B \bar{\partial}^*_B \beta = -\bar{\partial}_B \bar{\partial}_B \alpha = -F^{0,2}_B \alpha + \frac{1}{4}(\partial_B \alpha) \circ N_J,$$

where N_J is the Nijenhuis tensor of J. After some tedious estimates, based on the Weitzenböck formula (which may be found in Kotschick's article [K]), we infer that there exists some positive constant λ_0, which depends only on N_J, such that for all $\lambda \geq \lambda_0$ the following estimate holds

$$\varepsilon||d_B \alpha||^2 + \lambda||\bar{\alpha}\beta||^2 + C\lambda||\beta||^2 + \lambda||4\pi - |\alpha|^2||^2 \leq 16\pi^2 c_1(E) \cdot [\omega] \tag{4.3.7}$$

where $||\cdot|| := ||\cdot||_{L^2}$, $\varepsilon > 0$, and $C > 0$ are some constants. This inequality implies a necessary solvability condition:

$$c_1(E) \cdot [\omega] \geq 0. \tag{4.3.8}$$

Note that in the Kähler case this condition is a corollary of the existence of $\bar{\partial}_B$-holomorphic section φ_0 of E.

First, we consider the case when $c_1(E) \cdot [\omega] = 0$. In this case there exists a section α such that $|\alpha| \equiv 2\sqrt{\pi}$, thus E is necessary trivial. Then the SW_λ-equations have a trivial solution:

$$B \equiv 0, \;\; \alpha \equiv 2\sqrt{\pi}, \;\; \beta \equiv 0$$

and it can be shown that it is unique (up to gauge)(see [T3]). Hence in this case:

$$\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1.$$

Next we consider the case when $c_1(E) \cdot [\omega] > 0$. If we suppose that $\mathrm{SW}(X, \Gamma_E) \neq 0$, then, using (4.3.7), we can prove that the inequality

$$0 \leq c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega]$$

is necessary for the solvability of SW_λ-equations. Note that the equality in the left $\leq$-sign holds only for E trivial and in the right $\leq$-sign only for $E = K$.

Figure 4.2: the solvability diagram (Kähler case)

Figure 4.3: the solvability diagram (symplectic case)

4.4 From Seiberg-Witten equations to pseudoholomorphic curves

4.4.1 Seiberg-Witten equations

Let (X, ω) be a compact symplectic 4-manifold with a generic compatible almost complex structure J and $b_2^+ > 1$. Let E be a Hermitian line bundle over X with a Hermitian connection B. Suppose that X is provided with a Spin^c-structure (W_E, Γ_E), corresponding to E, and the Spin^c-connection, determined by B. We consider the SW$_\eta$-equations for this Spin^c-structure and

$$\eta = -F^+_{A_{\mathrm{can}}} + \frac{i\lambda}{4}\omega \quad , \quad \lambda > 0,$$

for the normalized sections

$$\alpha = \frac{\varphi_0}{\sqrt{\lambda}} \in \Omega^0(X, E) \quad , \quad \beta = \frac{\varphi_2}{\sqrt{\lambda}} \in \Omega^{0,2}(X, E).$$

These equations (called again the SW$_\lambda$-equations) have the form

$$\bar{\partial}_B \alpha + \bar{\partial}^*_B \beta = 0, \tag{4.4.1}$$

$$\frac{2}{\lambda}F_B^{0,2} = \bar{\alpha}\beta, \tag{4.4.2}$$

$$\frac{4i}{\lambda}F_B^{\omega} = 1 + |\beta|^2 - |\alpha|^2. \tag{4.4.3}$$

In this section we shall present a direct Taubes' construction, which associates with a λ-dependent family of solutions of SW$_\lambda$-equations for $\lambda \to \infty$ a pseudoholomorphic curve C in X with homology class $[C]$, Poincaré dual to $c_1(E)$. It is a non-trivial extension to symplectic 4-manifolds of the description of the moduli space of SW$_\lambda$-solutions on a compact Kähler surface in terms of effective divisors.

The following theorem is proved by Taubes [T5].

Theorem 4.4.1 (Taubes). *If $SW(X, \Gamma_E) \neq 0$ and $c_1(E) \cdot [\omega] > 0$, then there exists a (compact) pseudoholomorphic curve C, embedded into X, with the homology class $[C]$, which is Poincaré dual to $c_1(E)$.*

Remark 4.4.2. The pseudoholomorphic curve C, mentioned in the theorem, may not be connected, more precisely, $C = \sum_{j=1}^{k} d_j C_j$, where C_j are mutually disjoint, connected, pseudoholomorphic curves. We suppose, for simplicity, that $k = 1$ below.

4.4.2 A priori estimates

If SW$(X, \Gamma_E) \neq 0$, then the SW$_\lambda$-equations have a solution $(B_\lambda, (\alpha_\lambda, \beta_\lambda))$ for all $\lambda > 0$. The following a priori estimates for these solutions can be proved, using the Weitzenböck formula and the maximum principle.

$$|\alpha_\lambda| \leq 1 + \frac{C_1}{\lambda}, \tag{4.4.4}$$

$$|\beta_\lambda|^2 \leq \frac{C_2}{\lambda}(1 - |\alpha|^2) + \frac{C_3}{\lambda^3}, \tag{4.4.5}$$

$$||\bar{\partial}_{B_\lambda}\alpha_\lambda||^2 + ||d_{B_\lambda}\beta_\lambda||^2 \leq \frac{C_4}{\lambda}, \tag{4.4.6}$$

$$2\pi c_1(E) \cdot [\omega] - \frac{C_5}{\lambda} \leq \frac{\lambda}{2}\int_X |1 - |\alpha_\lambda|^2| \, d\,\mathrm{vol} \leq 2\pi c_1(E) \cdot [\omega] + \frac{C_5}{\lambda}, \tag{4.4.7}$$

$$|F^{\pm}_{B_\lambda}| \le C_6\lambda(1-|\alpha_\lambda|^2)+C_7, \tag{4.4.8}$$

where $C_1, \cdots, C_7$ are some constants, depending only on $c_1(E)$ and Riemannian metric.

These estimates imply that for $\lambda \to \infty$ we have: $|\alpha_\lambda| \to 1$ almost everywhere on X (away from the zeros of α_λ). Moreover, $||\bar\partial_{B_\lambda}\alpha_\lambda|| \to 0$, i.e. α_λ tends to become a $\bar\partial_{B_\lambda}$-holomorphic section of E. At the same time, $\beta_\lambda \to 0$ everywhere (together with its first derivatives). So the situation becomes more and more similar to the Kähler one for $\lambda \to \infty$.

4.4.3 Construction of a pseudoholomorphic curve

We denote by $C_\lambda := \alpha_\lambda^{-1}(0)$ the zero set of α_λ. The weak limit of this zero sets is the desired pseudoholomorphic curve C.

More precisely, we associate with an SW_λ-solution $(B_\lambda, (\alpha_\lambda, \beta_\lambda))$ a current:

$$F_\lambda(\eta) := \frac{i}{2\pi}\int_X F_{B_\lambda} \wedge \eta \tag{4.4.9}$$

for $\eta \in \Omega^2(X, \mathbb{R})$. The norms of F_λ, which are equal to

$$||F_\lambda|| = \sup_{0\neq\eta\in\Omega^2} \frac{|F_\lambda(\eta)|}{\sup_{x\in X}|\eta(x)|} \,,$$

are uniformly bounded, since (4.4.7) and (4.4.8) imply

$$||F_\lambda|| \le \frac{1}{2\pi}||F_{B_\lambda}||_{L^1} < C,$$

where $C > 0$ is a constant, which does not depend on λ. So we can find a sequence $\lambda_n \to \infty$ such that F_{λ_n} converges weakly to $\mathcal{F}$, which is a closed positive integral $(1,1)$-current, which is Poincaré dual of $c_1(E)$. The support of $\mathcal{F}$ is the desired pseudoholomorphic curve C.

4.4.4 The Seiberg-Witten equations on $\mathbb{R}^4$

Consider now the SW_1-equations on $X = \mathbb{R}^4$, provided with the standard Euclidean metric g_0 and standard symplectic form ω_0. They will play the role of a local model for SW_λ-equations on (X, ω, J) for $\lambda \to \infty$. We identify $(\mathbb{R}^4, \omega_0, J_0)$ with $\mathbb{C}^2$ and consider the trivial bundle E over $\mathbb{C}^2$. In this situation, SW_1-equations are written in the form

$$\bar\partial_B \alpha = 0, \tag{4.4.10}$$

$$F_B^{0,2} = 0, \tag{4.4.11}$$

$$4iF_B^{\omega} = 1 - |\alpha|^2. \tag{4.4.12}$$

Solutions of these equations satisfy the following a priori estimates:

$$|\alpha| \leq 1 \ , \ |\nabla_B \alpha| \leq C(1 - |\alpha|^2), \tag{4.4.13}$$

$$|F_B^-| \leq |F_B^+| = \frac{1}{4}(1 - |\alpha|^2), \tag{4.4.14}$$

$$\int_{B_R} (1 - |\alpha|^2)\, d\,\mathrm{vol} \leq CR^2 \tag{4.4.15}$$

for any ball $B_R = B_R(0)$ of radius R, and

$$\int \{|F_B^+|^2 - |F_B^-|^2\}\, d\,\mathrm{vol} \leq C < \infty, \tag{4.4.16}$$

From these inequalities we can deduce the following properties of solutions.

Property 4.4.3. *Either $|\alpha| \equiv 1$, or $|\alpha| < 1$ everywhere on $\mathbb{C}^2$. Moreover,*

- *if $|\alpha| \equiv 1$, then any solution is gauge equivalent to the trivial one, i.e. $B \equiv 0$, $\alpha \equiv 1$.*

- *if $|\alpha| < 1$, then the zero set $\alpha^{-1}(0)$ coincides with the zero set of a complex polynomial on $\mathbb{C}^2$, the degree of which is controlled by the constant C in* (4.4.13), (4.4.14), (4.4.15) *and* (4.4.16).

Property 4.4.4. *Either $|F_B^-| \equiv |F_B^+|$, or $|F_B^-| < |F_B^+|$ everywhere on $\mathbb{C}^2$. If $|F_B^-| \equiv |F_B^-|$, then there exists a $\mathbb{C}$-linear projection:*

$$\pi : \mathbb{C}^2 \to \mathbb{C}$$

such that (B, α) is gauge equivalent to $\pi^*(B_1, \alpha_1)$, where (B_1, α_1) is a vortex solution on $\mathbb{C}$ with finite energy.

Property 4.4.5.

$$\frac{1}{4\pi} \int \{|F_B^+|^2 - |F_B^-|^2\} d\,\mathrm{vol}\,(X)$$

is a non-negative integer.

Property 4.4.6. *$1 - |\alpha|^2$ and $|d_B \alpha|^2$ decrease exponentially fast with the distance from $\alpha^{-1}(0)$.*

4.4.5 Reduction to the local model

Let (X, ω, J) be a compact symplectic four-manifold, provided with the compatible Riemannian metric g. For arbitrary $x_0 \in X$ we can define a Gaussian coordinate chart at x_0, which is an embedding

$$h : \mathbf{R}^4 \hookrightarrow X,$$

sending $0 \mapsto x_0$, such that

$$h^* g = g_0 + O(|y|^2) \quad , \quad h^* \omega = \omega_0 + O(|y|).$$

Here $(\mathbf{R}^4, g_0, \omega_0)$ is the standard Euclidean four-space with coordinates $y = (y_i)_{i=1}^4$.

Suppose that

$$(B, (\alpha, \beta)) := (B_\lambda, (\alpha_\lambda, \beta_\lambda))$$

is a SW_λ-solution on (X, g, ω). Then $h^*(B, (\alpha, \beta))$ are SW_λ-data on $\mathbb{R}^4$. Applying the dilation

$$\delta_\lambda : y \mapsto \frac{y}{\sqrt{\lambda}}$$

(which is analogous to introducing the "slow time" variable), we obtain

$$(\underline{B}, (\underline{\alpha}, \underline{\beta})) = \delta_\lambda^* h^*(B, (\alpha, \beta)).$$

These are the SW_1-data on $(\mathbb{R}^4, \underline{g}, \underline{\omega})$, where

$$|\underline{g} - g_0| \leq \frac{C}{\lambda}|y|^2 \quad , \quad |\underline{\omega} - \omega_0| \leq \frac{C}{\sqrt{\lambda}}|y|$$

on the "big" ball of radius $\sqrt{\lambda}$ (i.e. for $|y| \leq \sqrt{\lambda}$).

The data $(\underline{B}, (\underline{\alpha}, \underline{\beta}))$ satisfy the SW_1-equations on $(\mathbb{R}^4, \underline{g}, \underline{\omega})$ and are estimated on the "small" ball of radius $1/\sqrt{\lambda}$ through $h^*(B, (\alpha, \beta))$ on $(\mathbb{R}^4, \underline{g}, \underline{\omega})$ by the following inequalities:

$$|\underline{\alpha}(y)| = \left|\alpha\left(\frac{y}{\sqrt{\lambda}}\right)\right| \leq C \quad , \quad |d_{\underline{B}}\underline{\alpha}(y)| = \frac{1}{\sqrt{\lambda}}\left|d_B\alpha\left(\frac{y}{\sqrt{\lambda}}\right)\right| \leq C, \quad (4.4.17)$$

$$|\underline{\beta}| = \frac{1}{\lambda}\left|\beta\left(\frac{y}{\sqrt{\lambda}}\right)\right| \leq \frac{C}{(\sqrt{\lambda})^3} \quad , \quad |d_{\underline{B}}\underline{\beta}(y)| = \left(\frac{1}{\sqrt{\lambda}}\right)^3\left|d_B\beta\left(\frac{y}{\sqrt{\lambda}}\right)\right| \leq \frac{C}{\lambda}, \quad (4.4.18)$$

$$|F_{\underline{B}}(y)| = \frac{1}{\lambda}\left|F_B\left(\frac{y}{\sqrt{\lambda}}\right)\right| \leq C. \quad (4.4.19)$$

Here the norm is taken with respect to the standard flat metric of $\mathbb{R}^4$.

4.4.6 Compactness lemma and existence of vortex-like solutions

The following lemma is due to Taubes.

Lemma 4.4.7 (Compactness lemma). *We assume that $\{\lambda_n\}$ is an unbounded sequence, i.e. $\lambda_n \to \infty$ as $n \to \infty$, and let $(B_n, (\alpha_n, \beta_n))$ be the corresponding sequence of SW_{λ_n}-data on (X, g, ω). Let $\{x_n\}$ be an arbitrary sequence of points in X, $\{h_n\}$ the corresponding sequence of Gaussian charts at x_n, and $(\underline{B}_n, (\underline{\alpha}_n, \underline{\beta}_n))$ — the SW_1-solutions on $(\mathbb{R}^4, \underline{g}_n, \underline{\omega}_n)$, constructed from $(B_n, (\alpha_n, \beta_n))$ (with the help of Gaussian charts $\{h_n\}$ at $\{x_n\}$), as in the previous subsection. Then there exists a subsequence of $[\underline{B}_n, (\underline{\alpha}_n, \underline{\beta}_n)]$ (where $[\underline{B}_n, (\underline{\alpha}_n, \underline{\beta}_n)]$ denotes the gauge equivalence class of $(\underline{B}_n, (\underline{\alpha}_n, \underline{\beta}_n))$), converging in the C^∞-topology on compact subsets of $\mathbb{R}^4$ to some SW_1-solution $(B_0, (\alpha_0, 0)) =: (B_0, \alpha_0)$ on $(\mathbb{R}^4, g_0, \omega_0)$. This solution satisfies all estimates* (4.4.13), (4.4.14), (4.4.15) *and* (4.4.16) *on* $\mathbb{R}^4$.

One can try to apply this lemma to study the pseudoholomorphic curve C, defined above. Let $x_0 \in C \subset X$, and apply Lemma 4.4.7 to $\lambda_n \to \infty$ and $x_n \equiv x_0$. Then, according to the lemma, the corresponding sequence $[\underline{B}_n, (\underline{\alpha}_n, \underline{\beta}_n)]$ will have a subsequence, converging (on compact subsets of $\mathbb{R}^4$) to an SW_1-solution (B_0, α_0) on $\mathbb{R}^4$. By the contruction, this solution should not depend on the radius in the spherical coordinates on $\mathbb{R}^4$. It means that for such a solution

$$|F^+_{B_0}| \equiv |F^-_{B_0}|$$

and so, according to the Property 4.4.4, (B_0, α_0) is a vortex-like solution.

While heuristically evident, this argument is hard to justify directly. There are two ways to prove the existence of a vortex-like solution, centered at a given point $x_0 \in C$.

One way is to use the argument, similar to that of Kronheimer and Mrowka [?]. We denote by $Y := S^1 \times C$ a spherical tubular neighborhood of C, provided with the product metric. Suppose that the metric of X is the product metric of $[-\epsilon, \epsilon] \times Y$ in a neighborhood of Y. Denote by (X_R, g_R) the Riemannian manifold, obtained from X by cutting along Y and inserting the cylinder $[-R, R] \times Y$. We identify X_R with X (note that g_0 coincides with the original metric of X). Then the SW_λ-equations on X can be considered as the gradient equation on X_R with $R = \lambda$, governed by a Chern-Simons functional F on Y. More precisely, an SW_λ-solution $(B, (\alpha, \beta))$ is represented (in the radial gauge) by a path $\gamma(r)$ in the configuration space of 3-dimensional data on Y and the SW_λ-equations for $\gamma(r)$ are equivalent to the gradient equation for $\gamma(r)$, governed by $F(\gamma(r))$:

$$\frac{d}{dr}\gamma(r) = \nabla F(\gamma(r)).$$

It follows that $F(\gamma(r))$ is non-decreasing in r and it may be proved that the total variation of $F(\gamma(r))$ is bounded from above by a constant, not depending on R. Consider now solutions on the cylinder $T := [0, 1] \times Y$. Then for each N we can find a solution of the equations on $T_N := [N, N+1] \times Y$, identified with T, with the gradient less than $1/N$. So we have a sequence of solutions on T with gradient, converging to zero. Taking a converging subsequence, we shall obtain in the limit a solution of SW_1-equations on T, which does not depend on r. Extending it to a translation-invariant solution on $\mathbb{R} \times Y$, we get a solution $\gamma(r)$ of SW_1-equations on $\mathbb{R} \times Y$, which does not depend on r. For such a solution, $|F^+_{B_0}| \equiv |F^-_{B_0}|$.

Another way to prove the existence of vortex-like solutions is to use the following localization lemma, due to Taubes.

4.4.7 Localization lemma

Lemma 4.4.8 (Localization lemma). *Fix $\varepsilon > 0$, $\delta > 0$, $R \geq 1$, $k \in \mathbb{N}$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, and for any SW_λ-solution $(B, (\alpha, \beta))$ on (X, g, ω) the following is true:*

1. *For a fixed $x \in X$:*

(a) *For any Gaussian chart h at x, we construct, as above, an SW_1-solution $(\underline{B}, (\underline{\alpha}, \underline{\beta}))$ on $(\mathbb{R}^4, \underline{g}, \underline{\omega})$. Then there exists an SW_1-solution (B_0, α_0) on $(\mathbb{R}^4, g_0, \omega_0)$ such that the distance between $[\underline{B}, (\underline{\alpha}, \underline{\beta})]$ and $[B_0, \alpha_0]$, measured on the ball $B_R \subset \mathbb{R}^4$ in C^k-norm, is less than ε.*

(b) *(B_0, α_0) satisfies (4.4.13), (4.4.14), (4.4.15) and (4.4.16) with a constant C, depending only on g and $c_1(E)$ (but not on λ and $(B, (\alpha, \beta))$).*

(c) *$(B_0, \alpha_0) \equiv (B_0^x, \alpha_0^x)$ depends on x, and $\alpha_0 \not\equiv const$ if $|\alpha(x)| < 1$.*

2. *For varying $x \in X$:*

(a) *There exists a constant C, depending only on g and $c_1(E)$, such that the set:*

$$\Omega_\delta := \{x \in X : \int_{B_R} \{|F^+_{B_0^x}|^2 - |F^-_{B_0^x}|^2\} \, d\,\mathrm{vol}(g_0) \geq \delta\}$$

may be covered by less than C/δ balls of radius $R/\sqrt{\lambda}$ (that is, Ω_δ is a poor set).

Using this Lemma, we can find a sequence $x_n \to x_0 \in C$ and an associated sequence of $\delta_n \to 0$ such that

$$\int_{B_R} \{|F^+_{\underline{B_n}}|^2 - |F^-_{\underline{B_n}}|^2\} \, d\,\mathrm{vol}(g_0) \; < \delta_n.$$

Then by the Compactness Lemma we can find a subsequence of $[\underline{B_n}, (\underline{\alpha_n}, \underline{\beta_n})]$, converging on B_R to a SW_1-solution (B_0, α_0) on $\mathbb{R}^4$, so that

$$||F^+_{B_0}||_{L^2(B_R)} = ||F^-_{B_0}||_{L^2(B_R)}.$$

Since $|F^+_{B_0}| > |F^-_{B_0}|$, it follows that $|F^+_{B_0}| \equiv |F^-_{B_0}|$ on B_R. Hence, (B_0, α_0) is a vortex-like solution.

Given a point $x_0 \in C$, we can find a sequence of SW_{λ_n}-solutions $(B_n, (\alpha_n, \beta_n))$, converging to a vortex-like solution (B_0, α_0), centered at x_0. In this sense, SW_λ-equations on X for $\lambda \to \infty$ reduce to a family of vortex equations on the normal bundle $N \to C$.

4.5 From pseudoholomorphic curves to Seiberg-Witten equations

4.5.1 Neighborhood geometry of a pseudoholomorphic curve

Let C be a compact pseudoholomorphic curve in a compact symplectic four-manifold (X, ω, J, g). We also assume that C is connected and smooth. Let

$$\pi : N \to C$$

be the normal bundle of C, where N_x for $x \in C$ is identified with the orthogonal complement of T_xC in T_xX. Since J preserves T_xC, it also preserves N_x. Thus $\pi : N \to C$ is a complex line bundle. We introduce a fiber-constant almost complex structure on N by

$$J_0 := \pi^*(J|_{TC}).$$

Also the metric g on X induces a Riemannian metric on N with the Levi-Civita connection θ, and we denote $d := d_\theta$, $\bar\partial := \bar\partial_{J_0}$ on C. We fix a J_0-holomorphic fiber coordinate s on N such that $ds = d_\theta s$ is a J_0-holomorphic 1-form on π-fibers.

Let $U \to C$ be a disk subbundle of $N \to C$, formed by disks of sufficiently small radius (we set it equal to one, for simplicity). Then by the map $\exp : U \to X$, we can identify U with a tubular neighborhood $\exp(U)$ of C in X and define another almost complex structure

$$J := \exp^{-1} J|_{\exp U}$$

on N.

The holomorphic tangent bundle $T^{1,0}_J U$ with respect to this almost complex structure is generated locally by a unitary frame $\{\kappa_1, \kappa_2\}$, where κ_1 is a section of $\pi^*(T^*C)\otimes\mathbb{C}$, generating $T^{1,0}_x C$ for x in C, and $\kappa_2 := ds+\sigma+O(|s|^2)$ is a section of $\pi^*(T^*C)\otimes N\otimes\mathbb{C}$. In the definition of κ_2, ds is a J_0-holomorphic term, σ is a non-holomorphic linear perturbation. More precisely,

$$\sigma = \sigma^{1,0} + \sigma^{0,1}$$

with respect to J_0 so that

$$\sigma^{1,0} = -\bar{\nu} s + \gamma \bar{s} \quad , \quad \sigma^{0,1} = \nu s + \mu \bar{s},$$

where a (0,1)-form ν is a section of $\pi^*(T^*C)\otimes\mathbb{C}$, a (0,1)-form μ is a section of $\pi^*(T^*C)\otimes\pi^*N^2\otimes\mathbb{C}$, and a (1,0)-form γ is a section of $\pi^*(T^*C)\otimes\pi^*N^2\otimes\mathbb{C}$. Thus, σ measures the linear variation of the almost complex structure J_0. Note that

$$\omega = i(\kappa_1 \wedge \bar{\kappa}_1 + \kappa_2 \wedge \bar{\kappa}_2).$$

4.5.2 Vortex bundle

Recall that the d-vortex moduli space $\mathfrak{M}_d$ in Section 1.2 is identified with $\mathbb{C}^d$ by the map

$$S : \mathbb{C}^d \to \Omega^1_s \times H_s.$$

This map associates with any $y = (y_1, y_2, \cdots, y_d)$ a pair (A, Φ) with $A = i(\bar{\partial} - \partial)u$, $\Phi = P_y e^{-u}$, where

$$P_y(z) = z^d + y_1 z^{d-1} + \cdots + y_d,$$

and u is a unique solution of the Liouville-type equation

$$4i\partial\bar{\partial}u = *\frac{1}{2}(1 - |P_y|^2 e^{-2u}),$$

satisfying the asymptotic condition: $u(z) \sim d \cdot \log|z|)$ for $|z| \to \infty$.

There is a U(1)-action on $\mathfrak{M}_d$, generated by the natural U(1)-action on $\mathbb{C}^d$ via the identification S. The action on $\mathbb{C}^d$ is given by

$$(y_1, \cdots y_d) \mapsto (e^{i\theta} y_1, \cdots, e^{id\theta} y_d).$$

The corresponding U(1)-action on $\mathfrak{M}_d$ has the form

$$(A, \Phi) \mapsto (\delta^*_\theta A, e^{id\theta} \delta^*_\theta \Phi),$$

where $\delta_\theta : z \mapsto e^{i\theta} z$ is the rotation of $\mathbb{C}$.

There is also a scale action which is generated by the dilation

$$\rho_r : z \mapsto \sqrt{r} z$$

for $r > 0$. This dilation induces a map

$$\rho_r^* : (A, \Phi) \mapsto (A_r, \Phi_r),$$

sending a vortex solution (A, Φ) to a solution (A_r, Φ_r) of the scaled vortex equations:

$$\bar{\partial}_{A_r} \Phi_r = 0, \tag{4.5.1}$$

$$2i(dA_r) = r(1 - |\Phi_r|^2). \tag{4.5.2}$$

Now we return to (X, ω, J, g) and the pseudoholomorphic curve C. We denote by $L \to C$ the unit-circle subbundle of $\pi : N \to C$ with the natural U(1)-action and define the *d-vortex bundle*, associated with $\pi : N \to C$, as

$$\mathcal{L}_d := L \times_{\mathrm{U}(1)} \mathfrak{M}_d \to C \qquad (\mathfrak{M}_d\text{-picture}).$$

By identifying $\mathfrak{M}_d$ with $\mathbb{C}^d$, this is isomorphic to the bundle

$$\mathcal{N}_d := \bigoplus_{m=1}^{d} N^m \to C \qquad (\mathbb{C}^d\text{-picture}).$$

A *d-vortex section* τ, i.e. a section of $\mathcal{L}_d$, is represented by a family $\{\tau_x\}_{x \in C}$ of d-vortex solutions $\tau_x = [A_x, \Phi_x]$ on N_x, $x \in C$. In particular, the zero section τ_0 generates a family of the radial d-vortex solutions. Note that the scale ρ_r^*-action extends in a natural way to d-vortex sections.

4.5.3 Construction of Seiberg-Witten data from d-vortex sections

Let $U_\delta \to C$ be the disk (of radius $\delta < 1$) subbundle of $U \to C$. And let $\tau = [A, \Phi]$ be a given d-vortex section.

Then we can construct a bundle $E \to X$ by gluing the trivial bundle over $X \setminus U_{1/2}$ with $\pi^* N^d \to U$ with the help of the following gluing map

$$(U \setminus U_{1/2}) \times \mathbb{C} \to \pi^* N^d,$$

sending (x, ζ) to $(x, \zeta\Phi_r/|\Phi_r|)$, where the scale r is chosen so that $\{\text{zeros of } \Phi_r\} \subset U_{1/2}$. Then $c_1(E)$ is Poincaré dual to $d[C]$.

For the construction of $(B, (\alpha, \beta))$, we use a bump function χ_δ on X, which satisfy $\chi_\delta \equiv 1$ on U_δ and $\chi_\delta \equiv 0$ outside $U_{1/2}$. Then we can construct α by gluing $\alpha_r \equiv 1$ on $X \setminus U_{1/2}$ with $\alpha_r = \Phi_r/(\chi_\delta + (1-\chi_\delta)|\Phi_r|)$, and B by gluing the trivial connection on $X \setminus U_{1/2}$ with $B_r = \chi_\delta A_r + (1-\chi_\delta)\alpha_r^{-1}\nabla\alpha_r$. We call the constructed $(B_r(\alpha_r, 0))$ the *Seiberg-Witten data* on X.

Plugging these Seiberg-Witten data into the Seiberg-Witten equations, we obtain the following inequalities:

$$|D_{B_r}(\alpha_r, 0)| \leq Ce^{-c\sqrt{r}\cdot\text{dist}},$$

$$\left|F^+_{B_r} + \frac{ir}{4}(1 - |\alpha_r|^2)\omega\right| \leq C\sqrt{r}e^{-c\sqrt{r}\cdot\text{dist}},$$

where "dist" is the distance from C. These estimates are not satisfactory for r large. In order to obtain better estimates, we need to know how vortex sections behave in the direction of the curve C. By analogy with the adiabatic limit argument from Section 2.3, one can expect that in order to yield better estimates the vortex section should satisfy a kind of adiabatic equation, which we are going to derive in the rest of this section.

4.5.4 Derivation of adiabatic equation

First of all we rewrite the Seiberg-Witten equations in a neighborhood of C for the Seiberg-Witten data, constructed in the previous subsection. Note that in a neighborhood of C we have: $E = \pi^* N^d$, $\alpha_r = \Phi_r$ and $B_r = A_r$. Then the Seiberg-Witten equations

$$D_{B_r}(\alpha_r, 0) = 0,$$

$$F^+_{B_r} + \frac{ir}{4}(1 - |\alpha_r|^2)\omega = 0$$

near C take on the form:

$$\bar{\partial}_{J,A_r}\Phi_r = P_J^{0,1}(d_{A_r}\Phi_r) = 0, \tag{4.5.3}$$

$$P_J^+(dA_r) + \frac{ir}{4}(1 - |\Phi_r|^2)\omega = 0, \tag{4.5.4}$$

where we denote by $P_J^{0,1}$ (resp. P_J^+) the orthogonal projection of the bundle Λ^1 of 1-forms to the subbundle $\Lambda_J^{0,1}$ of $(0,1)$-forms with respect the almost complex structure J (resp. of the bundle Λ^2 of 2-forms to the subbundle Λ^2_+ of self-dual 2-forms).

Next we consider a perturbation $\tilde{\tau} = [\tilde{A}, \tilde{\Phi}]$ of the original d-vortex solution $\tau = [A, \Phi]$ of the form

$$\tilde{A} = A + \varepsilon a \quad , \quad \tilde{\Phi} = \Phi + \varepsilon \varphi$$

with $\varepsilon = 1/\sqrt{r}$, where $(a, \varphi) \perp T_{(A,\Phi)}\mathcal{M}_d$. In terms of the linearized vortex operator $\mathcal{D}_{(A,\Phi)}$ this orthogonality condition may be rewritten in the following way. Denote by $\{n_\mu\}$ an L^2-basis of $\ker \mathcal{D}_{(A,\Phi)}$. Then the above orthogonality condition is equivalent to the relation

$$\langle (a, \varphi), n_\mu \rangle = 0 \quad \text{for} \quad \mu = 1, 2, \cdots, 2d,$$

under the assumption that the gauge fixing condition is satisfied

$$\delta^*_{(A,\Phi)} n_\mu = 0 \quad , \quad \delta^*_{(A,\Phi)}(a, \varphi) = 0$$

(this is similar to the three-dimensional case).

Plugging $\tilde{\tau}$ into the Seiberg-Witten equations (4.5.3) and (4.5.4), we obtain

$$P^{0,1}_J(d_{A_r}\Phi_r) + \epsilon P^{0,1}_J(d_{A_r}\varphi_r + a_r\Phi_r) + \cdots = 0, \tag{4.5.5}$$

$$P^+_J(dA_r) + \frac{ir}{4}(1 - |\Phi_r|^2)\omega + \epsilon P^+_J(da_r) - \epsilon\frac{ir}{2}\mathrm{Re}(\varphi_r\bar{\Phi}_r)\omega + \cdots = 0, \tag{4.5.6}$$

where by dots we denote the higher order terms in ε.

We decompose all differential operators d in (4.5.5) and (4.5.6) at points $z \in C$ into the vertical d^V, i.e. normal (along N_z), and horizontal d^C, i.e. tangential (along T_zC), components: $d = d^V + d^C$. For d^V-derivations we can suppose that the gauge class $[A_z, \Phi_z]$ at $z \in C$ is fixed (since the change of the gauge class corresponds to the change inside $\mathfrak{M}_d$, i.e. in the base of $\mathcal{L}_d \to C$) but the almost complex structure J_z changes (since J_z depends on the fiber parameter s in N_z). For d^C we can suppose, on the contrary, that the almost complex structure J_z coincides with the (fiber-constant) almost complex structure $J_{0,z}$, but the gauge class $[A_z, \Phi_z]$ changes. Note that the same decomposition is valid for $\bar{\partial}_{A_r}$ and d_{A_r}.

We consider first the tangential component of Seiberg–Witten equations and start from the radial case, when τ is the zero section of $\mathcal{L}_d \to C$. In this case we can drop all d^C-derivatives of (A, Φ) (since the gauge class $[A, \Phi]$ is fixed). We start with the tangential (horizontal) component of

SW_r-equations (4.5.5) and (4.5.6) and collect all ε-terms (first-order terms in ε).

From (4.5.5) we obtain

$$P_J^{0,1}(d^C_{A_r}\varphi_r + a_r\Phi_r) = \bar\partial^C_{A_r}\varphi_r + a_r^{0,1}\Phi_r,$$

where we drop the subindex "J_0", when we take derivatives with respect to J_0. Note that the right hand side of the last equation coincides with the first component

$$\mathcal{D}^{(1)}_{(A,\Phi)}(a_r,\varphi_r) = \bar\partial^C_{A_r}\varphi_r + a_r^{0,1}\Phi_r$$

of the linearized vortex operator.

From (4.5.6) we obtain the following ε-term

$$P_J^+(d^C a_r) - \frac{ir}{2}\mathrm{Re}(\varphi_r\bar\Phi_r)\omega = d^C_+ a_r - \frac{ir}{2}\mathrm{Re}(\varphi_r\bar\Phi_r)\omega\ ,$$

coinciding with the second component $\mathcal{D}^{(2)}_{(A,\Phi)}(a_r,\varphi_r)$ of the linearized vortex operator.

So the ε-term in the tangential part of (4.5.5) and (4.5.6) has the form

$$\mathcal{D}_{(A,\Phi)}(a_r,\varphi_r)$$

in the radial case. In the general case, it takes the form

$$\mathcal{D}_{(A,\Phi)}(a_r,\varphi_r) + (\varphi_r^1, a_r^1)$$

where (φ_r^1, a_r^1) depends on the variation of the gauge class $[A,\Phi]$. For $\varepsilon = 1/\sqrt{r}$ we have $\varphi_r^1 = \sqrt{r}\bar\partial^C_{A_r}\Phi_r$, $a_r^1 = \sqrt{r}d^C_+A_r$.

Next we consider the normal component of SW_r-equations (4.5.5) and (4.1.2). We introduce the "slow" fiber parameter $\zeta = \varepsilon s$. Note that

$$P_J^{0,1}(d_{A_r}\Phi_r) = P_{J_0}^{0,1}(d_{A_r}\Phi_r) + \sigma^{0,1}[\partial^V_{A_r}\Phi_r] = \bar\partial^V_{A_r}\Phi_r + \sigma^{0,1}[\partial^V_{A_r}\Phi_r]\ .$$

Here, the 1-form σ measures the dependence of J on the fiber parameter s (in other words, σ is equal to $\delta_s J$). Its $(0,1)$-component (w.r. to J_0) may be written in the form

$$\sigma^{0,1} = \nu s + \mu\bar s$$

for suitable 1-forms ν, μ. We denote by $[\partial^V_{A_r}\Phi_r]$ the partial derivative of Φ_r in s, i.e.

$$\partial^V_{A_r}\Phi_r = [\partial^V_{A_r}\Phi_r]ds\ .$$

Note also that $\bar\partial^V_{A_r}\Phi_r$ vanishes because of the vortex equations for (A,Φ).

Then the normal component of (4.5.5) has the form

$$
\begin{aligned}
P_J^{0,1}(d^V_{A_r}\Phi_r)+&\varepsilon P_J^{0,1}(d^V_{A_r}\varphi_r)\\
&= \bar\partial^V_{A_r}\Phi_r + \sigma^{0,1}[\partial^V_{A_r}\Phi_r] + \varepsilon P_J^{0,1}(d^V_{A_r}\varphi_r)\\
&= \epsilon\sigma^{0,1}[\partial^V_{A_r}\Phi_r] + \dots ,
\end{aligned}
$$

where we have introduced the slow variable $\zeta = \varepsilon s$ in the last equality and denoted by dots the higher-order terms.

The normal component of (4.5.6) has the form

$$
\begin{aligned}
P_J^+(d^V A_r)+&\frac{ir}{4}(1-|\Phi_r|^2)\omega + \epsilon P_J^+(d^V a_r)\\
&= \left\{ d^V_+ A_r + \frac{ir}{4}(1-|\Phi_r|^2)\omega \right\} + \sigma^{0,1}\wedge d^V A_r + \epsilon P_J^+(d^V a_r).
\end{aligned}
\tag{4.5.7}
$$

The term in brackets vanishes because of the vortex equations. By introducing the slow variable ζ, we obtain

$$
\varepsilon\sigma^{0,1}\wedge d^V A_r + \dots .
$$

Hence, the ε-term in the normal part of (4.5.5) and (4.5.6) has the form

$$
\sigma^{0,1}\left([\partial^V_{\zeta,A_r}\Phi_r], d^V A_r\right) .
$$

Consequently, the total ε-term of (4.5.5) and (4.5.6) becomes

$$
\mathcal{D}_{(A,\Phi)}(a_r,\varphi_r) + \sigma^{0,1}\left([\partial^V_{\zeta,A_r}\Phi_r], d^V A_r\right) + (\varphi_r^1, a_r^1) . \tag{4.5.8}
$$

We shall obtain the adiabatic equation for $\tau_0 = [A,\Phi]$ by equalizing (4.5.8) to zero and using the orthogonality condition to eliminate the term containing (a_r,φ_r).

Since $\langle(a,\varphi), n_\mu\rangle = 0$, there exists a unique L^2-solution (b,ψ) of the equation

$$
\mathcal{D}^*_{(A,\Phi)}(b,\psi) = (a,\varphi)
$$

where $\mathcal{D}^*_{(A,\Phi)}$ is the L^2-adjoint of $\mathcal{D}_{(A,\Phi)}$. We introduce the linearized vortex Laplacian:

$$
L_{(A,\Phi)} := \mathcal{D}^*_{(A,\Phi)}\mathcal{D}_{(A,\Phi)} = \mathcal{D}_{(A,\Phi)}\mathcal{D}^*_{(A,\Phi)},
$$

and note that

$$
L_{(A,\Phi)} n_\mu = 0 \qquad \text{for} \quad \mu = 1,\cdots,2d.
$$

Therefore

$$\begin{aligned}\langle \mathcal{D}_{(A,\Phi)}(a_r,\varphi_r), n_\mu\rangle &= \langle L_{(A,\Phi)}(b_r,\psi_r), n_\mu\rangle \\ &= \langle (b_r,\psi_r), L_{(A_r,\Phi_r)} n_\mu\rangle \\ &= 0.\end{aligned}$$

We take now the inner product of (4.5.8), equalized to zero, with the zero modes n_μ and obtain the following equation for $\tau_0 = [A,\Phi]$:

$$\langle(\sigma^{0,1}[\partial_{\zeta,A_r}\Phi_r], d_\zeta A_r), n_\mu\rangle + \langle(\varphi_r^1, a_r^1), n_\mu\rangle = 0.$$

This is precisely the condition used by [T7]. We have obtained it here as an adiabatic equation.

4.5.5 The space of adiabatic sections

In this subsection, we consider the adiabatic equation:

$$\langle p(A_r,\Phi_r), n_\mu\rangle := \langle(\sigma^{0,1}[\partial_{\zeta,A_r}\Phi_r], d_\zeta A_r), n_\mu\rangle + \langle(\varphi_r^1, a_r^1), n_\mu\rangle = 0 \quad (4.5.9)$$

and list some properties of *adiabatic sections*, i.e. sections of the d-vortex bundle $\mathcal{L}_d \to C$, satisfying the adiabatic equation.

We denote the set of adiabatic sections of $\mathcal{L}_d \to C$ by $\mathcal{Z}$. Then $\mathcal{Z}$ is locally compact and its smooth part $\mathcal{Z}_{\mathrm{reg}}$, consisting of adiabatic sections, for which the adiabatic equation is satisfied transversally, is an oriented manifold of dimension

$$2d(1-g) + d(d+1)n,$$

where g is the genus of C and n is the degree of the map $\pi : N \to C$. By identifying $\mathfrak{M}_d$ with $\mathbb{C}^d$, we can identify $\mathcal{Z}$ with the space of sections of the bundle

$$\mathcal{N}_d = \bigoplus_{m=1}^{d} N^d \to C,$$

satisfying a non-linear $\bar\partial$-equation:

$$\bar\partial y + \nu\chi y + \mu F(y) = 0, \quad (4.5.10)$$

where $y = (y_1,\cdots,y_d)$ is a section of $\mathcal{N}_d$, χ is a homomorphism defined by

$$\chi : (y_1,\cdots y_d) \mapsto (y_1, 2y_2,\cdots, dy_d),$$

F is a smooth fiber operator on the section of $\mathcal{N}_d$, and ν, μ are the linear parts of the variation of the almost complex structure. When $d = 1$, we can write it down in terms of $\tau = [A, \Phi]$ as

$$\bar{\partial} f_\tau + \nu f_\tau + \mu \bar{f}_\tau = 0, \tag{4.5.11}$$

where f_τ is the section of $N \to C$ with the image $\Phi^{-1}(0)$. If, in particular, the latter set coincides with C, i.e. $\{\text{zeros of } \Phi\} = C$, then this $\bar{\partial}$-equation is equivalent to the pseudoholomorphicity of C.

4.5.6 Construction of Seiberg-Witten solutions from adiabatic sections

Let $\tau = [A, \Phi] \in \mathcal{L}$, so $p(A, \Phi) \perp T_{(A,\Phi)}\mathfrak{M}_d$. Then there exists a unique L^2-solution of the equation:

$$\mathcal{D}^*_{(A,\Phi)}(b, \psi) = p(A, \Phi).$$

We construct new Seiberg-Witten data $(\tilde{B}, (\tilde{\alpha}, \tilde{\beta}))$ by modifying the old data $(B, (\alpha, \beta))$ (E remains the same). These new data are obtained by gluing (over $U \setminus U_{1/2}$) the data $(B_r, (\alpha_r, 0))$ on $X \setminus U_{1/2}$ with $(B_r + \chi_\delta b_r, (\alpha_r, \chi_\delta \psi_r))$ on U. The new data satisfy the following estimates (cp. them with those in the preceding subsection):

$$|D_{\tilde{B}_r}(\tilde{\alpha}_r, \tilde{\beta}_r)| \leq \frac{C}{\sqrt{r}} e^{-c\sqrt{r}\cdot\mathrm{dist}}, \tag{4.5.12}$$

$$\left| F^+_{\tilde{B}_r} + \frac{ir}{4}(1 - |\tilde{\alpha}_r|^2 + |\tilde{\beta}_r|^2)\omega + \frac{r}{2}(\tilde{\alpha}_r\bar{\tilde{\beta}}_r - \bar{\tilde{\alpha}}_r\tilde{\beta}_r) \right| \leq Ce^{-c\sqrt{r}\cdot\mathrm{dist}}. \quad (4.5.13)$$

Using these estimates in the implicit function theorem, Taubes proves the following theorem (cf. [T7].

Theorem 4.5.1 (Taubes). *Let K be a relatively compact open subset in $\mathcal{Z}_{reg}$. Then for $r \geq r_0$ there exists a continuous map*

$$\Psi_r : K \to \mathcal{M}_r$$

defined by $\tau = [A, \Phi] \mapsto (\vec{B}_r, (\vec{\alpha}_r, \vec{\beta}_r))$, where

$$\vec{B}_r = \tilde{B}_r + \sqrt{r}B' \quad , \quad \vec{\alpha}_r = \tilde{\alpha}_r + \alpha' \quad , \quad \vec{\beta}_r = \tilde{\beta}_r + \beta'.$$

Furthermore, the error-term $\gamma' = (B', (\alpha', \beta'))$ satisfies the following inequalities:

$$||\nabla\gamma'||_{L^2} + \sqrt{r}||\gamma'||_{L^2} \leq \frac{C}{\sqrt{r}}, \quad (4.5.14)$$

$$\sup_X |\nabla\gamma'| + \sqrt{r}\sup_X |\gamma'| \leq C. \quad (4.5.15)$$

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