SEIBERG–WITTEN EQUATIONS AND COMPLEX ABRIKOSOV STRINGS

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Abstract. An interpretation of Abrikosov strings in the Abelian (2+1)-dimensional Higgs model in terms of the adiabatic limit is given. In analogous way, complex Abrikosov strings are interpreted in terms of the reduction of 4-dimensional Seiberg–Witten equations to pseudoholomorphic curves.

The Seiberg–Witten equations, found in 1994 (cf. [10,11,19]), were brought immediately into the center of activity of 4-dimensional topologists and mathematical physicists. The Seiberg–Witten equations, in contrast with the intensively studied during the last years instanton equations, are Abelian, and the moduli spaces of their solutions are compact. On the other hand, the Seiberg-Witten invariants, determined by these moduli spaces, provide the same information as the Donaldson invariants, drawn from the moduli spaces of solutions of instanton equations (cf. [8]). The indicated advantages of Seiberg–Witten invariants make it possible to apply them successfully to the solution of a number of well known problems in the topology of smooth 4-dimensional manifolds (cf. [1,4,15]).

The so called "Taubes equation" (cf. [16,17,18]):

$$\mathrm{SW} = \mathrm{Gr}\ ,$$

is one of the most remarkable results on the Seiberg-Wiiten equations. It means that the Gromov invariant of a 4-dimensional symplectic manifold (counting the number of pseudoholomorphic curves in a given homology class) is expressed by a simple formula through its Seiberg–Witten invariant. To prove his equation, Taubes proposed a procedure of the reduction of Seiberg–Witten equations, defined on a 4-dimensional symplectic manifold X, to pseudoholomorphic curves, which can be described as follows. The Seiberg–Witten equations, in contrast with the conformally invariant instanton equations, are not invariant under the scale transformations (and it is a kind of "payment" for the compactness of their moduli spaces). If we plug the scale parameter r into them and drive it to infinity, one of the components of their solution will vanish, while the zeros of the other component will approach a chain $C = \sum d_k C_k$, consisting of pseudoholomorphic curves C_k with multiplicities d_k. The original Seiberg–Witten equation will reduce respectively to a family of vortex equations, defined on the complex normal planes of the curves C_k. The chain $C = \sum d_k C_k$ may be considered as a complex analogue of the Abrikosov strings from the superconductivity theory. In the converse direction, one can construct from a given pseudoholomorphic chain $C = \sum d_k C_k$ on the manifold X and a family of solutions of vortex equations on the normal planes of C_k the data for the Seiberg–Witten equations on X. These data will satisfy with any prescribed accuracy the Seiberg–Witten equations for $r \to \infty$, if the given family of solutions (which may be identified with a pseudoholomorphic curve in the moduli space of solutions of vortex equations) satisfy a complex Abrikosov equation (of the type of a nonlinear $\bar{\partial}$-equation) on every C_k.

The paper is organized as follows. It starts from the dimension 2. In Ch.I some necessary notions on the vortex equation in the complex plane and its physical meaning are collected. In Ch.II we turn to the dimension 3 and consider the dynamical (2+1)-dimensional Abelian Higgs model, and also its Euclidean analogue. By passing to the adiabatic limit in these models, we reduce the problem of the approximate solution of the corresponding Euler–Lagrange equations to the solution of adiabatic Abrikosov equations on the moduli space of solutions of vortex equations (this idea in application to the Ginzburg–Landau equations was first precisely formulated and realized by Stuart in [14]). The adiabatic solutions in the case of dynamical Higgs model describe the scattering of "slowly moving" vortices, and in the case of its Euclidean analogue the Abrikosov strings, "slightly differing" from straight lines. In the last Ch.III we turn to the dimension 4 and Seiberg–Witten equations on compact symplectic manifolds. We describe here the aforementioned reduction procedure of these equations to pseudoholomorphic chains.

The main goal of this paper is to give a unified exposition of the two indicated constructions — the passing to the adiabatic limit in the (2+1)-dimensional Higgs model (and its Euclidean analogue) and the reduction of the 4-dimensional Seiberg–Witten equations to pseudoholomorphic curves. Both procedures allow to describe approximately solutions of the original equations in terms of curves on the moduli spaces of the reduced equations. Moreover, as it is shown in the paper, the reductiond procedure in the 4-dimensional case is nothing more than a complex analogue of the 3-dimensional adiabatic limit.

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Chapter I. DIMENSION TWO: THE VORTEX EQUATION

1.1. Ginzburg–Landau Equations

Superconductivity. For the temperature, close to the absolute zero, some metals and alloys start to behave themselves as superconductors. In other words, the electric current runs along them practically without resistance and evaporating any heat. According to the modern theory (cf., e.g., [6]), this phenomena is due to the fact that for very low temperatures free electrons may unite into the so called *Cooper pairs* forming quasiparticles with the doubled electric charge and zero spin. Thus these Cooper pairs are bosons and precisely their current is superconducting. Along with the superconductivity arising, the magnetic field is "pushed out" from the superconductor and this effect, called the Meissner effect, is one of the main practical criteria of the superconductivity.

If we start to increase the external magnetic field, then for some critical value the superconductivity will break down and the magnetic field will penetrate into the superconductor. This process can develop by two different scenarios and, accordingly, all superconductors are divided into two different classes. In superconductors of the Ist type (this class contains mostly superconducting metals) the breakdown of the superconductivity looks like a jump and occurs simultaneously in the whole interior of the superconductor. In superconductors of the IInd type (this class contains mostly superconducting alloys) this process proceeds gradually, by small discrete steps. When the external magnetic field exceeds the first critical value, tube-like zones, *flux tubes*, arise inside the superconductor which are directed along the external magnetic field. In the centre of these tubes, along the so called *Abrikosov strings* (or Abrikosov threads) we have the normal conductivity, inside the tubes it is of mixed type, and outside them the superconductivity is preserved. With the increasing of the external magnetic field the number of the tubes is also increased, and at the second critical value the superconductor transforms into the normal conductor.

Ginzburg–Landau equations. For the description of the superconductor, being in the stationary intermediate state, Ginzburg and Landau proposed a Lagrangian, which can be written in the following dimensionless form

$$\mathcal{L}(\mathbf{A},\Phi)=\frac{\mathbf{B}^2}{2}+|\nabla_{\mathbf{A}}\Phi|^2+\frac{\lambda}{4}(1-|\Phi|^2)^2 \tag{1.1}$$

where $\mathbf{A}$ is the vector-potential of the electromagnetic field, $\mathbf{B}=\nabla\times\mathbf{A}=\operatorname{rot}\mathbf{A}$ is the magnetic field (magnetic induction), $\nabla_{\mathbf{A}}=\nabla-i\mathbf{A}$ is the covariant derivative, Φ is the wave function of Cooper pairs, and $\lambda>0$ is a basic physical constant, equal to $2\delta^2/l^2$ where δ is a characteristic scale of magnetic field (the depth of its penetration into the superconductor), and l is a characterictis scale of Φ (the correlation radius).

Assuming the time fixed, we introduce the energy functional

$$E(\mathbf{A},\Phi)=\frac{1}{2}\int\mathcal{L}(\mathbf{A},\Phi)d^3x\ . \tag{1.2}$$

From its variation with respect to $\mathbf{A}$ and Φ (by omitting boundary terms) we get the Euler–Lagrange equations for E in the following form

$$\begin{cases}\nabla^2_{\mathbf{A}}\Phi+\dfrac{\lambda}{2}\Phi(1-|\Phi|^2)=0\\ \nabla^2\mathbf{A}-\mathbf{A}|\Phi|^2+\operatorname{Im}(\bar{\Phi}\nabla\Phi)=0\ .\end{cases} \tag{1.3}$$

Solutions of these equations describe the superconductor in the intermediate state.

The parameter λ has a deep physical meaning. For the superconductors of the Ist type $0<\lambda<1$, i.e. $\delta<l/\sqrt{2}$. This relation between the characteristic scales of the fields $\mathbf{B}$ and Φ prevent the appearing of the normal conductivity zones inside the superconductor. On the opposite, the relation $\lambda>1 \iff \delta>l/\sqrt{2}$, satisfied for the superconductors of the IInd type, makes their appearing energetically favourable.

Vortex lines. The magnetic flow

$$\int\limits_{\substack{\text{section}\\ \text{of the tube}}} B\, d\sigma \ ,$$

transferred by every flux tube, is an integer multiple of the elementary flux quantum, equal (in our dimensionless units) to π. The field $\mathbf{B}$ itself is directed along the tube axis, and its charateristic size is determined by δ. The wave function Φ vanishes in the centre of the tube, along the Abrikosov string (which corresponds to the normal conductivity), and outside the tube its modulus is equal to 1. The characterictic size of the zone, where $|\Phi| < 1$, is determined by l. The argument of Φ is changed by $2\pi \times \{\text{integer}\}$ after winding around the string. By writing Φ in the polar coordinates

$$\Phi = \rho e^{i\theta} \ ,$$

we see that the vector field $\mathbf{v} := \nabla\theta$, restricted to the 2-dimensional section orthogonal to the tube axis, looks like a hydrodynamical vortex. By this reason the Abrikosov strings are also called the *vortex lines*.

The physical parameter λ has the following interpretation in terms of vortex lines. For $\lambda < 1$ the vortex lines of the same vorticity (such lines have the flux tubes with the magnetic flow of the same sign) are attracted, while for $\lambda > 1$ they are repelled. For $\lambda = 1$ the intervortex interaction is absent and we can expect that for such value of λ there exist many static solutions with vortex lines of the same vorticity.

1.2. Vortex equations

The Euler–Lagrange equations. We consider a 2-dimensional analogue of the problem, stated in Sec.1.1 but formulated in more habitual mathematical terms. The Lagrangian (1.1) will have the form

$$\mathcal{L}(A,\Phi) = |F_A|^2 + |d_A\Phi|^2 + \frac{\lambda}{4}(|\Phi|^2 - 1)^2 \tag{1.4}$$

where A is a U(1)-connection on $\mathbb{R}^2$, which is given in the coordinates $x = (x_1, x_2)$ on $\mathbb{R}^2$ by a smooth 1-form

$$A = A_1 dx_1 + A_2 dx_2$$

where A_1, A_2 are smooth functions of $x = (x_1, x_2)$ with pure imaginary values, F_A is the curvature of A, given by a smooth 2-form

$$F_A = dA = F_{12} dx_1 \wedge dx_2$$

where $F_{12} = \partial_1 A_2 - \partial_2 A_1$ and $\partial_j = \partial/\partial x_j, \ j = 1, 2$. The exterior derivative d_A is equal to $d_A = d + A$, i.e.

$$d_A\Phi = (\partial_1 + A_1)\Phi dx_1 + (\partial_2 + A_2)\Phi dx_2 =: \sum_{j=1}^{2} \nabla_{A,j}\Phi dx_j \ .$$

The function $\Phi = \Phi_1 + i\Phi_2$ is a smooth complex-valued function on $\mathbb{R}^2$, and $\lambda > 0$ is a constant.

The finiteness of the energy

$$U(A,\Phi) = \frac{1}{2}\int \mathcal{L}(A,\Phi)d^2x \ < \ \infty \tag{1.5}$$

implies (cf. [2]) the following asymptotic condition

$$|\Phi(x)| \longrightarrow 1 \quad \text{for} \quad |x| \to \infty \ . \tag{1.6}$$

If this limit is uniform, then one can define a topological invariant of the model, called the *vortex number* d, which coincides with the winding number of Φ at infinity. In other words, for a sufficiently large R

$$d = \text{ the winding number of } \Phi : S^1_R \longrightarrow \{|\Phi| \approx 1\} = S^1 \ .$$

If we require, in addition, that the derivative $|d_A\Phi|$ decreases sufficiently fast at infinity (e.g., $|d_A\Phi| \leq \text{const}/|x|^{1+\delta}$ for some $\delta > 0$), then the vortex number d may be written also in the form

$$d = \frac{i}{2\pi}\int F_A \ . \tag{1.7}$$

The Euler–Lagrange equations for the functional $U(A,\Phi)$ have the form

$$\begin{cases} \partial_j F_{jk} = J_k \ , \quad j,k = 1,2 \\ \nabla^2_A\Phi = \dfrac{\lambda}{2}\Phi(|\Phi|^2 - 1) \end{cases} \tag{1.8}$$

where

$$\nabla_{A,j} = \partial_j + A_j \ , \ \nabla^2_A = \sum_{j=1}^{2} \nabla^2_{A,j} \ ,$$

and the current J_k is equal to

$$J_k = \mathrm{Im}(\Phi\overline{\nabla_{A,k}\Phi}) \ ,$$

or, in a more invariant form,

$$\begin{cases} d^*F_A + i\mathrm{Im}(\Phi\overline{d_A\Phi}) = 0 \\ \Delta_A\Phi = \dfrac{\lambda}{2}\Phi(1 - |\Phi|^2) \end{cases} \tag{1.9}$$

where $*$ is the Hodge operator for the standard Euclidean metric on $\mathbb{R}^2$, $d^* := - * d*$ is the adjoint operator of d in L^2-metric on $\mathbb{R}^2$ and, analogously, $d^*_A = - * d_A *$, so that $\Delta_A := d^*_A d_A$ is the covariant Laplace operator.

Vortex equations. We turn now to the critical case $\lambda = 1$ and introduce the complex coordinate $z = x_1 + ix_2$ on the plane $\mathbb{R}^2$. The covariant Cauchy–Riemann operator is, by definition, equal to

$$\bar{\partial}_A = \bar{\partial}_z + A^{0,1}$$

where $\bar{\partial}_z = \partial/\partial\bar{z}$, and $A = A^{1,0} + A^{0,1}$ is the decomposition of the form A into the components of the form

$$A = A^{1,0} + A^{0,1} = A_{1,0}dz + A_{0,1}d\bar{z} \ , \quad A^{1,0} = -\overline{A^{0,1}} \ .$$

In the same way, $\partial_A = \partial_z + A^{1,0}$ where $\partial_z = \partial/\partial z$.

The energy functional $U(A, \Phi)$ for $\lambda = 1$ may be transformed into the following form (*Bogomolny transform*)

$$U(A,\Phi) = \frac{1}{2}\int \{2|\bar{\partial}_A\Phi|^2 + |iF_{12} + \frac{1}{2}(|\Phi|^2 - 1)|^2\}\frac{i}{2}dz \wedge d\bar{z} + \frac{i}{2}\int F_A \ . \qquad (1.10)$$

The last term is topological, since

$$\frac{i}{2}\int F_A = \pi d \ .$$

Hence, if $d \geq 0$, then it follows from (1.10) that

$$U(A,\Phi) \geq \pi d \ , \qquad (1.11)$$

and the equality is possible if and only if the following *vortex equations* are satisfied

$$\left\{ \begin{array}{l} \bar{\partial}_A\Phi = 0 \\ iF_{12} = \frac{1}{2}(1 - |\Phi|^2) \iff iF_A = \frac{1}{2} * (1 - |\Phi|^2) \ . \end{array} \right. \qquad (1.12)$$

For $d < 0$ one can use another Bogomolny transform

$$U(A,\Phi) = \frac{1}{2}\int \{2|\partial_A\Phi|^2 + |iF_{12} - \frac{1}{2}(|\Phi|^2 - 1)|^2\}\frac{i}{2}dz \wedge d\bar{z} - \frac{i}{2}\int F_A \ ,$$

which implies again that

$$U(A,\Phi) \geq -\pi d \ ,$$

and the equality is attained on solutions of the *antivortex equations*

$$\left\{ \begin{array}{l} \partial_A\Phi = 0 \\ iF_{12} = \frac{1}{2}(|\Phi|^2 - 1) \iff iF_A = \frac{1}{2} * (|\Phi|^2 - 1) \ . \end{array} \right. \qquad (1.13)$$

Solutions of the vortex equations (1.12) with the vortex number $d \geq 0$ and finite energy are called the *d-vortices*. It follows from the above that their energy is necessarily equal to $U(A,\Phi) = \pi d$. Solutions of the antivortex equations (1.13) with the vortex number $d < 0$ and finite energy (necessarily equal to $U(A,\Phi) = -\pi d$) are called the *$|d|$-antivortices*.

The equations (1.12), (1.13), as well as the original Lagrangian (1.4), are invariant under the gauge transformations of the form

$$A \longmapsto A + id\chi \quad , \quad \Phi \longmapsto e^{-i\chi}\Phi \qquad (1.14)$$

where χ is a smooth real-valued function on $\mathbb{R}^2 = \mathbb{C}$.

1.3. The theorems of Taubes

Formulations of the theorems. A description of all solutions of the vortex equations modulo gauge equivalence is given by the following Taubes' theorems (cf. [2]).

THEOREM 1 ($d \geq 0$ CASE). *For a given $d \geq 0$ and an arbitrary collection of different points $z_1, \ldots, z_k$ of the complex plane with multiplicities $d_1, \ldots, d_k$, such that*

$$\sum_{j=1}^{k} d_j = d \ ,$$

there exists a unique (up to gauge equivalence) solution (A, Φ) of the vortex equations (1.12), having the finite energy $U(A, \Phi)$ (equal to πd), for which

(1) *the zeros of Φ coincide with $z_1, \ldots, z_k$ and have the given multiplicities $d_1, \ldots, d_k$ in these points, so that in a neighbourhood of any point z_j the function Φ may be represented in the form*

$$\Phi(z) = \varphi_j(z)(z - z_j)^{d_j}$$

where φ_j is a smooth function in a neighbourhood of z_j, having no zeros in this neighbourhood;

(2) *the vortex number of the solution is equal to*

$$\frac{i}{2\pi} \int F_A = \sum_{j=1}^{k} d_j = d \ .$$

In the case $d < 0$ there is an analogous theorem on solving the antivortex equations (1.13). It may be deduced from the Theorem 1, using the fact that the substitution $A \mapsto \bar{A}$, $\Phi \mapsto \bar{\Phi}$ sends solutions of the vortex equations (1.12) to solutions of the antivortex equations (1.13) and vice versa.

The case $d = 0$ is singled out. In this case there exists an evident solution of the equations (1.12) of the form $\Phi \equiv 1$, $A \equiv 0$. It follows from the Theorem 1 that any other solution with $d = 0$ and finite energy is gauge equivalent to this one.

THEOREM 2. *Any critical point (A, Φ) of the energy functional $U(A, \Phi)$ with $U(A, \Phi) < \infty$ for $\lambda = 1$, i.e. any solution of the Euler–Lagrange equations (1.8) with finite energy is gauge equivalent either to a d-vortex from the Theorem 1 (for $d \geq 0$) or to its analogue — $|d|$-antivortex (for $d < 0$).*

The Theorem 2 implies, in particular, that the Euler–Lagrange equations for $\lambda = 1$ cannot possess a solution of the type "vortex – antivortex", having finite energy. In other words, in the considered physical model, described by the equations (1.8) for $\lambda = 1$, all static configurations are stable and realize minima of the energy $U(A, \Phi)$ (in a given topological class).

The Theorem 2 admits also another interpretation, according to which the Euler–Lagrange equations (1.8) for $\lambda = 1$, which are the 2nd order equations in (A, Φ), (under the finite energy condition) are equivalent either to the vortex equations

(1.12) for $d \geq 0$ or to the antivortex equations (1.13) for $d < 0$. Note that the equations (1.12), as well as the equations (1.13), have the 1st order in (A, Φ). Such equivalence of the Euler–Lagrange equations to the 1st order equations occurs seldom in gauge field theories. More often the Euler–Lagrange equations for critical points of the action are not equivalent to the equations for its minima. In other words, in such models there exist non-minimal (i.e. physically unstable) solutions. We have such a situation for the Bogomolny equations for the Prasad–Sommerfield monopoles on $\mathbb{R}^3$ and the duality equations for the instantons on $\mathbb{R}^4$.

The Theorems 1 and 2 give a description of all solutions of the vortex equations (with finite energy) up to gauge equivalence. We introduce the *moduli space* of solutions of (1.12) with finite energy and vortex number $d > 0$ by setting

$$\mathcal{M}_d = \frac{\{\text{solutions } (A, \Phi) \text{ of vortex equations with vortex number } d\}}{\{\text{gauge transforms}\}} \ . \tag{1.15}$$

It follows from Taubes' theorems that $\mathcal{M}_d$ coincides with the dth symmetric power of the complex plane $\mathbb{C}$, i.e.

$$\mathcal{M}_d = S^d\mathbb{C} \cong \mathbb{C}^d \ . \tag{1.16}$$

Idea of the proof of Theorem 1. A solution (A, Φ) of the vortex equations with given zeros of Φ is constructed in two steps. As the first step we produce a "tentative solution" (Ansatz), given by a pair (A_0, Φ_0). This pair satisfies the 1st vortex equation and has the finite energy $U(A_0, \Phi_0)$, while the function Φ_0 has given zeros and satisfies the asymptotic condition (1.6). On the second stage a small correction to the above Ansatz is found by substituting of (A_0, Φ_0) into the 2nd vortex equation and solving the obtained non-linear equation with the help of an appropriate version of the fixed point theorem.

For the construction of the Ansatz (A_0, Φ_0) we use the fact that for large $|z|$ we have $|\Phi(z)| \sim 1$ and the vortex equation (1.12) may be considered approximately linear, i.e. subject to the superposition principle. Hence we can take for (A_0, Φ_0) the superposition of the radial solutions with the unique zero at z_j, $j = 1, \ldots, k$.

We turn to the more detailed exposition. Write the desired function Φ in the form

$$\Phi = e^{\frac{u+i\theta}{2}} \tag{1.17}$$

where u and θ have real values on $\mathbb{C}$. Since the function Φ must have zeros of multiplicity d_j at z_j, $j = 1, \ldots, k$, it follows that for $z \to z_j$ we must have $u(z) \to -\infty$, and θ must be a multi-valued function of z with branching points at z_j of order d_j. The functions u and θ are supposed to be smooth outside $z_1, \ldots, z_k$. The 1st equation (1.12) implies that

$$\bar\partial\Phi = -A^{0,1}\Phi \quad \Longrightarrow \quad A^{0,1} = -\bar\partial \ln \Phi$$

outside the zeros of Φ. Because of the smoothness of the desired form $A^{0,1}$, this relation is fulfilled by continuity also at zeros of Φ. Hence

$$A^{0,1} = -\frac{1}{2}\bar\partial(u + i\theta) \ , \ A^{1,0} = -\overline{A^{0,1}} = -\frac{1}{2}\bar\partial(u - i\theta) \ . \tag{1.18}$$

By fixing the function θ, we can get rid of the gauge ambiguity in the definition of (A, Φ). (Note that the poles of u and the branching points of θ do not change under gauge transforms). Take for θ the multi-valued function

$$\theta_0 = 2\sum_{j=1}^{k} d_j \operatorname{Arg}(z - z_j) \tag{1.19}$$

and plug (A, Φ), defined by the formulae (1.17),(1.18) into the 2nd vortex equation (1.12). We get the equation on u of the form

$$\Delta u = e^u - 1 + 4\pi \sum_{j=1}^{k} d_j \delta(z - z_j) \tag{1.20}$$

with the asymptotic condition

$$u(z) \longrightarrow \quad \text{for} \quad |z| \to \infty \tag{1.21}$$

which follows from the condition: $|\Phi(z)| \to 1$ for $|z| \to \infty$.

We introduce the function

$$u_0(z) = -\sum_{j=1}^{k} \ln\left(1 + \frac{\mu}{|z - z_j|^2}\right)^{d_j} \quad \text{where} \quad \mu > 4d\ . \tag{1.22}$$

This function satisfies (in the sense of distributions) to the Laplace equation of the form

$$\Delta u_0 = 4\pi \sum_{j=1}^{k} d_j \delta(z - z_j) - 4\sum_{j=1}^{k} \frac{\mu d_j}{(\mu + |z - z_j|^2)^2}\ .$$

Hence, the function $v := u - u_0$ will satisfy the equation

$$\Delta v = he^v - 1 + g \tag{1.23}$$

where

$$h = e^{u_0} \quad , \quad g(z) = 4\sum_{j=1}^{k} \frac{\mu d_j}{(\mu + |z - z_j|^2)^2}\ ,$$

and the asymptotic condition

$$v(z) \longrightarrow 0 \quad \text{for} \quad |z| \to \infty\ . \tag{1.24}$$

Note that $h(z) > 0$, and $0 < g(z) < 1$ for all z. Moreover, the functions g and h are smooth.

The nonlinear equation of Liouville type (1.23) with the condition (1.24) has a unique solution, which is real analytic. It follows that the pair (A, Φ), given by the formulae (1.17),(1.18), where $u = u_0 + i\theta$, $\theta = \theta_0$, is a smooth solution of the vortex equations, having all properties listed in the formulation of the Theorem 1.

The solvability of the equation (1.23) is a corollary of a theorem on the solvability of the Liouville type equation

$$\Delta v = f e^{v} + g \ , \tag{1.25}$$

proved by Kazdan and Warner (cf. [3]). The equation (1.25) arises in the differential geometry in connection to the following problem:

find for a given Riemannian metric g with the Gaussian curvature k a conformally equivalent Riemannian metric G with the given Gaussian curvature K.

If we look for the metric G of the form $G = e^{2v}g$, then we get for the desired function v an equation of the Liouville type

$$\Delta_g v = K e^{2v} - k \ .$$

The solvability properties of this equation depend crucially on the signs of the functions k and K. In particular, it has a unique solution, if k and K are everywhere positive, as in the equation (1.23).

Chapter II. DIMENSION THREE: ABELIAN (2+1)-DIMENSIONAL HIGGS MODEL AND ABRIKOSOV STRINGS

2.1. Abelian Higgs model

The action. We turn to the dimension 3 by adding to the variables $(x_1, x_2) \in \mathbb{R}^2$ one more variable $x_0 = t$, which we identify with the time. In other words, we study the dynamical model in the space $\mathbb{R}^{2+1}$ which is described by the action functional

$$S(A, \Phi) = \int \{T(A, \Phi) - U(A, \Phi)\}\, dt \tag{2.1}$$

where the potential energy $U(A, \Phi)$ is equal, as before, to

$$U(A, \Phi) = \frac{1}{2} \int \left\{ |F_A|^2 + |d_A \Phi|^2 + \frac{\lambda}{4}(|\Phi|^2 - 1)^2 \right\} d^2x \ , \tag{2.2}$$

and the kinetic energy $T(A, \Phi)$ is given by the formula

$$T(A, \Phi) = \frac{1}{2} \int \left\{ \sum_{j=1}^{2} |F_{0j}|^2 + |d_{A,0}\Phi|^2 \right\} d^2x \ . \tag{2.3}$$

Here A is, as before, a U(1)-connection on $\mathbb{R}^{2+1}$, given by the 1-form

$$A = A_0 dt + A_1 dx_1 + A_2 dx_2$$

with smooth pure imaginary coefficients $A_\mu = A_\mu(t, x_1, x_2)$, $\mu = 0, 1, 2$, and F_A is the curvature of the connection A, given by the smooth 2-form

$$F_A = dA = \sum_{\mu,\nu=0}^{2} F_{\mu\nu} dx_\mu \wedge dx_\nu \quad , \quad F_{\mu\nu} = \partial_m u A_\nu - \partial_\nu A_\mu \ , \ \mu, \nu = 0, 1, 2 \ .$$

The covariant derivative $d_A = d + A$, and $d_{A,0}$ is its time component, so that

$$d_{A,0}\Phi = \partial_t \Phi dt + A_0 \Phi dt \ ,$$

$\Phi = \Phi(t, x_1, x_2)$ is a smooth complex-valued function on $\mathbb{R}^{2+1}$.

The Euler–Lagrange equations. The Euler–Lagrange equations for the action $S(A,\Phi)$ have the form

$$\begin{cases} \partial_0 F_{0j} + \epsilon_{jk}\partial_k F_{12} = i\mathrm{Im}(\bar{\Phi}\nabla_{A,j}\Phi) \ , \ j=1,2 \\ (\nabla^2_{A,0} - \nabla^2_{A,1} - \nabla^2_{A,2})\Phi = \frac{\lambda}{2}\Phi(1-|\Phi|^2) \\ \partial_1 F_{01} + \partial_2 F_{02} = i\mathrm{Im}(\bar{\Phi}\nabla_{A,0}\Phi) \end{cases} \tag{2.4}$$

where $\nabla_{A,\mu} = \partial_\mu + A_\mu$, $\mu = 0,1,2$, and $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. We are interested in solutions of this hyperbolic system of equations modulo gauge transforms

$$A_\mu \longmapsto A_\mu + i\partial_\mu\chi \ , \ \Phi \longmapsto e^{-i\chi}\Phi \tag{2.5}$$

where χ is a smooth real-valued function on $\mathbb{R}^{2+1}$.

The equations (2.4) become more tractable if we accept the time gauge, i.e. if we set $A_0 = 0$ (which can be always attained with the help of an appropriate gauge transform (2.5)). In this case the formula (2.2) for the kinetic energy may be rewritten in the form

$$T(A,\Phi) = \frac{1}{2}\left(\|\dot{A}\|^2_{L^2} + \|\dot{\Phi}\|^2_{L^2}\right)$$

where we denote by "dot" the time derivative, and by $\|\cdot\|_{L^2}$ the L^2-norm on $\mathbb{R}^2$. The Euler–Lagrange equations in this gauge take the form

$$\begin{cases} \ddot{A}_j + \epsilon_{jk}\partial_k F_{12} = i\mathrm{Im}(\bar{\Phi}\nabla_{A,j}\Phi) \ , \ j=1,2 \\ \ddot{\Phi} - \nabla^2_{A,1}\Phi - \nabla^2_{A,2}\Phi = \frac{\lambda}{2}\Phi(1-|\Phi|^2) \\ \partial_1\dot{A}_1 + \partial_2\dot{A}_2 = i\mathrm{Im}(\bar{\Phi}\dot{\Phi}) \ . \end{cases} \tag{2.6}$$

The 3rd equation is a kind of a constraint, which is imposed on the initial conditions at the time moment $t = 0$.

2.2. Heuristic considerations. Linearized vortex equations

Manton's approach. We are interested in solutions of the system (2.6) modulo gauge transforms. Such solutions will be considered as paths

$$t \longmapsto [A(t),\Phi(t)]$$

in the configuration space $\mathfrak{N}$ of smooth pairs $(A,\Phi) = (A(x_1,x_2),\Phi(x_1,x_2))$, defined modulo gauge transforms. Assuming that the vortex number of $[A(t),\Phi(t)]$ does not depend on t and is equal to $d > 0$, we get a path $t \mapsto [A(t),\Phi(t)]$ in the space

$$\mathfrak{N}_d = \frac{\{\text{smooth pairs } (A,\Phi) \text{ with vortex number } d\}}{\{\text{gauge transforms}\}} \ .$$

Manton has proposed (cf. [7]) the following heuristic method of the approximate description of solutions of the dynamical equations (2.6) for $\lambda = 1$ in terms of the moduli space $\mathcal{M}_d$ of solutions of (static) vortex equations (1.12) on $\mathbb{R}^2$. Suppose that the kinetic energy $T(A,\Phi)$ of the path $\gamma : t \mapsto [A(t),\Phi(t)]$ is small. It means that the path γ in the configuration space $\mathfrak{N}_d$ is close to the space of static solutions $\mathcal{M}_d$, and the less is its kinetic energy the closer is γ to $\mathcal{M}_d$. Assume that $T(A,\Phi) < \epsilon$ and introduce a new "slow" time variable $\tau = \epsilon t$ on γ. Then in the limit $\epsilon \to 0$ we obtain a curve γ_0 on $\mathcal{M}_d$, which should be, according to Manton, a geodesic on the space $\mathcal{M}_d$ in the metric, determined by the kinetic energy. So geodesics on $\mathcal{M}_d$ in T-metric describe approximately "slow moving" dynamical solutions of equations (2.6).

Linearized vortex equations. For a more formal presentation of Manton's idea we have to study carefully the structure of the tangent bundle of $\mathcal{M}_d$. We introduce for that the Sobolev version of $\mathcal{M}_d$, namely, we fix $s \geq 1$ and consider the space $\mathcal{V}_d^s$ of solutions (A,Φ) of vortex equations

$$\begin{cases} \bar{\partial}_A \Phi = 0 \\ iF_A = *\dfrac{1}{2}(1-|\Phi|^2) \end{cases} \tag{2.7}$$

on $\mathbb{R}^2$ where A is a 1-form on $\mathbb{R}^2$ with the coefficients, belonging to the Sobolev space $H_s(\mathbb{R}^2, i\mathbb{R})$, i.e.

$$A \in \Omega^1(\mathbb{R}^2) \otimes H_s(\mathbb{R}^2, i\mathbb{R}) =: \Omega^1_s \ ,$$

and Φ belongs to the Sobolev space $H_s(\mathbb{R}^2,\mathbb{C}) =: H_s$, i.e.

$$(A,\Phi) \in \Omega^1_s \times H_s \ .$$

The Sobolev version of the moduli space $\mathcal{M}_d$ is the space

$$\mathcal{M}_d^s = \frac{\mathcal{V}_d^s}{\mathcal{G}_{s+1}}$$

where $\mathcal{G}_{s+1}$ is the group of gauge transforms

$$A \longmapsto A + id\chi \quad , \quad \Phi \longmapsto e^{-i\chi}\Phi \ ,$$

given by functions $\chi \in H_{s+1}(\mathbb{R}^2,\mathbb{R})$. For $s \geq 1$ the space $\mathcal{M}_d^s = S^d\mathbb{C}$, i.e. does not depend on s, so we shall omit the index s where it cannot cause misunderstandings.

By varying the equations (2.7) with respect to A and Φ, we obtain the linearized vortex equations at a point (A,Φ), being a solution of (2.7):

$$\begin{cases} \bar{\partial}_A \varphi + a^{0,1}\Phi = 0 \\ *ida + \mathrm{Re}(\varphi\bar{\Phi}) = 0 \end{cases} \tag{2.8}$$

where $(a,\varphi) \in \Omega^1_s \times H_s$. The left hand side of these equations defines the linearized vortex operator

$$\mathcal{D}_{(A,\Phi)} : (a,\varphi) \longmapsto (\bar{\partial}_A\varphi + a^{0,1}\Phi, *ida + \mathrm{Re}(\varphi\bar{\Phi})) \ , \tag{2.9}$$

acting from the space $\Omega^1_s \times H_s$ into the space $\Omega^{0,1}_{s-1}(\mathbb{R}^2,\mathbb{C}) \times H_{s-1}(\mathbb{R}^2,\mathbb{R})$. Thus, the tangent bundle of the infinite-dimensional Hilbert space $\mathcal{V}_d$ is identified with the kernel of the operator $\mathcal{D}_{(A,\Phi)}$:

$$T_{(A,\Phi)}\mathcal{V}_d = \mathrm{Ker}\,\mathcal{D}_{(A,\Phi)} = \{(a,\varphi) \in \Omega^1_s \times H_s : \mathcal{D}_{(A,\Phi)}(a,\varphi) = 0\} \ . \tag{2.10}$$

The tangent space of the moduli space. The space $T_{(A,\Phi)}\mathcal{V}_d$ is invariant under the action of the infinitesimal gauge transforms of the form

$$a \longmapsto a + id\chi \quad , \quad \varphi \longmapsto \varphi - i\Phi\chi \tag{2.11}$$

where $\chi \in H_{s+1}(\mathbb{R}^2, \mathbb{R})$. So, the tangent space to the orbit of the algebra $\mathrm{Lie}\,\mathcal{G}$ of gauge transforms at the origin is identified with the space of pairs $(id\chi, -i\phi\chi) \in \Omega^1_s \times H_s$. If we introduce the tangent operator of the form

$$\delta_{(A,\Phi)} : \chi \longmapsto (id\chi, -i\Phi\chi) \ , \ H_{s+1}(\mathbb{R}^2, \mathbb{R}) \longrightarrow \Omega^1_s \times H_s \ ,$$

then its adjoint operator

$$\delta^*_{(A,\Phi)} : \Omega^1_s \times H_s \longrightarrow H_s(\mathbb{R}^2, \mathbb{R})$$

will have the form

$$\delta^*_{(A,\Phi)} : (a, \varphi) \longmapsto id^*a + i\mathrm{Im}(\bar{\Phi}\varphi) \ .$$

Since

$$\Omega^1_s \times H_s = T_{(A,\Phi)}\left(\mathcal{G} \cdot (A, \Phi)\right) \oplus \mathrm{Ker}\,\delta^*_{(A,\Phi)} \ ,$$

we can fix the infinitesimal gauge by the condition

$$\delta^*_{(A,\Phi)}(a, \varphi) = 0 \ . \tag{2.12}$$

Hence,

$$\begin{aligned} T_{(A,\Phi)}\mathcal{M}_d = \mathrm{Ker}\,\mathcal{D}_{(A,\Phi)} \bigcap \mathrm{Ker}\,\delta^*_{(A,\Phi)} = \\ = \{(a, \varphi) \in \Omega^1_s \times H_s : \mathcal{D}_{(A,\Phi)}(a, \varphi) = 0 \ , \ \delta^*_{(A,\Phi)}(a, \varphi) = 0\} \ . \end{aligned}$$

2.3. Adiabatic limit

The least squares method. We return to the statement of the problem from Sec.2.2. Let $t \mapsto q(t)$ be a path in $S^d\mathbb{C} = \mathbb{C}^d$. By Theorem 1 from Sec.1.3 we can associate with every $q \in S^d\mathbb{C}$ a unique (up to gauge) d-vortex solution $[A(q), \Phi(q)] \in \mathcal{M}_d$. So with any path $t \mapsto q(t)$ we associate the path

$$\gamma : t \longmapsto [A(q(t)), \Phi(q(t))]$$

in the space $\mathcal{M}_d$. It's convenient to assume that γ is a path in the space $\mathcal{V}_d$ of vortex solutions, satisfying to the gauge condition

$$\delta^*_{(A,\Phi)}(\dot{A}, \dot{\Phi}) = 0 \tag{2.13}$$

for every t. Consider a variation $\tilde{\gamma} : t \mapsto [\tilde{A}(t), \tilde{\Phi}(t)]$ of the path γ in the configuration space $\mathfrak{N}_d$ where

$$\tilde{A}(t) = A(q(t)) + a(t) \ , \ \tilde{\Phi}(t) = \Phi(q(t)) + \varphi(t) \ . \tag{2.14}$$

We look for a condition on the path $t \mapsto q(t)$, which is necessary for $\tilde{\gamma} = [\tilde{A}, \tilde{\Phi}]$ to be a solution of the dynamical equations (2.6), assuming at the same time that $\tilde{\gamma} = [\tilde{A}, \tilde{\Phi}]$ satisfies the gauge condition (2.13). Then it's natural to require the perturbation (a, φ) to be orthogonal to $T_{(A,\Phi)}\mathcal{M}_d$, excluding in this way the deformations in directions, tangent to $\mathcal{M}_d$. The condition of orthogonality of (a, φ) to $T_{(A,\Phi)}\mathcal{M}_d$ may be also obtained by the following argument. If $\tilde{\gamma} = [\tilde{A}, \tilde{\Phi}]$ is a solution of the dynamical equations (2.6), then the path $t \mapsto q(t)$ and, consequently, the representation (2.14) may be reconstructed from $\tilde{\gamma} = [\tilde{A}, \tilde{\Phi}]$ by the least squares method. In other words, the desired path $t \mapsto q(t)$ in $\mathcal{M}_d$ should minimize the value of

$$\int \left\{ |\tilde{A}(t) - A(q(t))|^2 + |\tilde{\Phi}(t) - \Phi(q(t))|^2 \right\} d^2x = \|\tilde{A} - A\|^2_{L^2} + \|\tilde{\Phi} - \Phi\|^2_{L^2} \ .$$

The extremals of this functional satisfy the Euler–Lagrange equation

$$\langle a, \delta A \rangle + \langle \varphi, \delta\Phi \rangle = 0 \ ,$$

which implies that (a, φ) should be orthogonal to $T_{(A,\Phi)}\mathcal{M}_d$, since the variations $(\delta A, \delta\Phi)$ with respect to q belong to $T_{(A,\Phi)}\mathcal{M}_d$. Assuming that (a, φ) satisfies also the gauge condition (2.12), we can rewrite the orthogonality condition $(a, \varphi) \perp T_{(A,\Phi)}\mathcal{M}_d$ in the form

$$(a, \varphi) \perp \operatorname{Ker} \mathcal{D}_{(A,\Phi)} \ . \tag{2.15}$$

If we have (for every t) a base of L^2-solutions $\{n_\mu\}$ of the equation

$$\mathcal{D}_{(A,\Phi)} n_\mu = 0 \quad , \ \mu = 1, \ldots, 2d \ , \tag{2.16}$$

then the condition (2.15) may be rewritten in the form

$$\langle (a, \varphi), n_\mu \rangle = 0 \quad , \ \mu = 1, \ldots, 2d \ . \tag{2.17}$$

Adiabatic equation. We introduce now a small parameter $\epsilon > 0$, replacing the decomposition (2.14) by the decomposition of the form

$$\tilde{A}(t) = A(q(t)) + \epsilon^2 a(t) \ , \ \tilde{\Phi}(t) = \Phi(q(t)) + \epsilon^2 \varphi(t) \ , \tag{2.18}$$

and switch on the "slow" time $\tau = \epsilon t$. Substituting $(\tilde{A}, \tilde{\Phi})$ into the dynamical equations (2.6), we write them in the form

$$\partial_t^2 (a, \varphi) + \mathcal{D}^*_{(A,\Phi)} \mathcal{D}_{(A,\Phi)} (a, \varphi) = \\ = \left(-\partial_\tau^2 A, -\partial_\tau^2 \Phi + \frac{\lambda - 1}{2} \Phi (1 - |\Phi|^2) \right) + \epsilon j \tag{2.19}$$

where j is the sum of nonlinear terms like the currents from the right hand sides of the equations (2.6) and so on (in the derivation of the formula (2.19) we have used the fact that $(A(q(t)), \Phi(q(t)))$ satisfy the vortex equations and, consequently, the equations (1.8) for $\lambda = 1$).

Differentiating the condition (2.17) twice by t, we obtain

$$\langle \partial_t^2 (a,\varphi), n_\mu\rangle = -\langle (a,\varphi), \partial_t^2 n_\mu\rangle - 2\langle \partial_t (a,\varphi), \partial_t n_\mu\rangle \ , \ \mu = 1,\dots,2d \ . \tag{2.20}$$

We plug into this relation the expression for $\partial_t^2(a,\varphi)$ from (2.19) and use the equation $\mathcal{D}_{(A,\Phi)} n_\mu = 0$. Then we get

$$\left\langle \left(\partial_\tau^2 A, \partial_\tau^2 \Phi - \frac{\lambda - 1}{2} \Phi (1 - |\Phi|^2) \right), n_\mu \right\rangle = \epsilon h \ , \ \mu = 1,\dots,2d \ , \tag{2.21}$$

where h is the sum of nonlinear terms from (2.19) and terms, containing derivatives of n_μ with respect to τ. By dropping out the terms in (2.21), depending on ϵ, we obtain the desired equation for the path γ, having the form

$$\left\langle \left(\partial_\tau^2 A, \partial_\tau^2 \Phi - \frac{\lambda - 1}{2} \Phi (1 - |\Phi|^2) \right), n_\mu \right\rangle = 0 \ , \ \mu = 1,\dots,2d \ . \tag{2.22}$$

Hence, the path γ, produced from the dynamical solution, given by the path $\tilde{\gamma}$, in the "slow" time limit for $\epsilon \to 0$ should satisfy the equation (2.22). Such a limit is called adiabatic in mechanics and theoretical physics, therefore we call the condition (2.22) the *adiabatic equation.*

The justification of the Manton's approach. We shall give below another derivation of the adiabatic equation, and now we note that for $\lambda = 1$ it is reduced to the condition

$$\langle \partial_\tau^2 (A,\Phi), n_\mu\rangle = 0 \ , \ \mu = 1,\dots,2d \ . \tag{2.23}$$

We show that this equation coincides with the geodesic equation for the T-metric, i.e. the heuristic Manton's principle is true indeed.

By definition, geodesics of the T-metric are the extremals of the functional

$$\int T(A,\Phi) d\tau = \frac{1}{2} \int \left\{ \|\dot{A}\|_{L^2}^2 + \|\dot{\Phi}\|_{L^2}^2 \right\} d\tau \ ,$$

restricted to the paths $\tau \mapsto [A(\tau), \Phi(\tau)]$, lying in $\mathcal{M}_d$. These extremals satisfy the Euler–Lagrange equation of the form

$$\int \left\{ \langle \dot{A}, \delta\dot{A}\rangle + \langle \dot{\Phi}, \delta\dot{\Phi}\rangle \right\} d\tau = - \int \left\{ \langle \ddot{A}, \delta A\rangle + \langle \ddot{\Phi}, \delta\Phi\rangle \right\} d\tau = 0 \ .$$

It implies that under the gauge condition the following relation holds

$$\langle \partial_\tau^2 (A,\Phi), n_\mu\rangle = 0 \ , \ \mu = 1,\dots,2d \ ,$$

i.e. the equation (2.23) is indeed the geodesic equation for the T-metric.

Adiabatic paths. The above derivation of the condition (2.23) suggests that also in the general case (i.e. for $\lambda \neq 1$) one can obtain the adiabatic equation (2.22) as a condition on extremals of the action functional, restricted to the paths, lying in $\mathcal{M}_d$. In more detail, we call the *path*

$$\tau \longmapsto [A(\tau), \Phi(\tau)] \tag{2.24}$$

in $\mathcal{M}_d$ *adiabatic*, if it is extremal for the action functional $S(A, \Phi)$, given by the formula (2.1), restricted to the paths of the form (2.24), lying in $\mathcal{M}_d$.

We show that adiabatic paths satisfy the adiabatic equation (2.22). The action $S(A, \Phi)$ has the form

$$S(A, \Phi) = \int \{T(A, \Phi) - U(A, \Phi)\}\, d\tau$$

where

$$T(A, \Phi) = \frac{1}{2}\left(\|\dot{A}\|^2_{L^2} + \|\dot{\Phi}\|^2_{L^2}\right) ,$$

and

$$U(A, \Phi) = \frac{1}{2}\left(\|dA\|^2_{L^2} + \|d_A\Phi\|^2_{L^2} + \frac{\lambda}{4}\|1 - |\Phi|^2\|^2_{L^2}\right) .$$

Its variation is equal to

$$\begin{aligned}
\delta T(A, \Phi) &= -\langle \ddot{A}, \delta A\rangle - \langle \ddot{\Phi}, \delta\Phi\rangle \\
\delta U(A, \Phi) &= \langle d^* dA + i\mathrm{Im}(\bar{\Phi} d_A \Phi), \delta A\rangle + \langle d^*_A d_A \Phi - \frac{\lambda}{2}\Phi(1 - |\Phi|^2), \delta\Phi\rangle .
\end{aligned}$$

Since the pair (A, Φ) satisfies the vortex equations for any τ (hence, the equations (1.8) for $\lambda = 1$), the Euler–Lagrange equation $\delta S(A, \Phi) = 0$ signifies that

$$\left(-\ddot{A}, -\ddot{\Phi} + \frac{\lambda - 1}{2}\Phi(1 - |\Phi|^2)\right) \perp T_{(A,\Phi)}\mathcal{M}_d ,$$

i.e. that

$$\left\langle \left(-\ddot{A}, -\ddot{\Phi} + \frac{\lambda - 1}{2}\Phi(1 - |\Phi|^2)\right), n_\mu \right\rangle = 0 , \ \mu = 1, \dots, 2d .$$

2.4. Adiabatic Hamiltonian equations

Adiabatic Hamiltonian. The adiabatic equation (2.22) has the Newtonian form

$$\langle \partial^2_\tau (A, \Phi), n_\mu\rangle = \frac{\lambda - 1}{2}\langle \Phi(1 - |\Phi|^2), n_\mu\rangle \ \mu = 1, \dots, 2d ,$$

where one can consider the left hand side as the "acceleration×mass", and the right hand side as the "force". It is an indication that the equations (2.22) are, in fact, the Hamiltonian equations on the cotangent bundle $T^*\mathcal{M}_d$. The corresponding *adiabatic Hamiltonian* has the form

$$H_{\mathrm{ad}} = T_{\mathrm{ad}} + U_{\mathrm{ad}} \tag{2.25}$$

where

$$U_{\rm ad}(A,\Phi)=\frac{\lambda-1}{8}\int(1-|\Phi|^2)^2d^2x\ . \tag{2.26}$$

We write down the Hamiltonian equations, given by the adiabatic Hamiltonian in local coordinates on $T^*\mathcal{M}_d$. Let $\{q_\mu\}$, $\mu=1,\dots,2d$, be local coordinates on $\mathcal{M}_d$ in a neighbourhood of a point $q=[A,\Phi]$ (in our case such coordinates exist even globally on $\mathcal{M}_d=S^d\mathbb{C}=\mathbb{C}^d$). Let $\{\dot q_\mu\}$, $\mu=1,\dots,2d$, denote, as usual, the fibre coordinates on $T_q\mathcal{M}_d$, and $\{n_\mu\}$, $\mu=1,\dots,2d$, denote, as before, a base of L^2-solutions of the linearized equation

$$\mathcal{D}_q n_\mu=0\ ,$$

subject to the gauge condition: $\delta_q^* n_\mu=0$. Then the T-metric on $T_q\mathcal{M}_d$ is given by the formula

$$T_q(\dot q,\dot q)=\sum_{\mu,\nu=1}^{2d}\langle n_\mu,n_\nu\rangle\dot q_\mu\dot q_\nu\ .$$

Let momenta $\{p_\mu\}$, $\mu=1,\dots,2d$ denote, as usual, the fibre coordinates on $T_q^*\mathcal{M}_d$, so that

$$p_\mu=\sum_{\nu=1}^{2d}\langle n_\mu,n_\nu\rangle\dot q_\nu \tag{2.27}$$

and $T_q^*\mathcal{M}_d$ is provided with the dual T-metric, i.e. $T_q(p,p):=T_q(\dot q,\dot q)$ where p_μ and $\dot q_\mu$ are related by (2.27). Then the adiabatic Hamiltonian will be given by the formula

$$H_{\rm ad}(p,q)=\frac{1}{2}T_q(p,p)+U_{\rm ad}(q)\ ,$$

and the corresponding adiabatic Hamiltonian equations have the form

$$\begin{cases}\dfrac{dp_\mu}{d\tau}=-\dfrac{\partial H_{\rm ad}}{\partial q_\mu} & \text{(Newton law)}\\[2mm] \dfrac{dq_\mu}{d\tau}=\dfrac{\partial H_{\rm ad}}{\partial p_\mu} & \text{(definition of } p_\mu)\end{cases} \tag{2.28}$$

where $\mu=1,\dots,2d$.

The theorem on the solvability. Having in mind the above considerations, one should expect that any solution of the adiabatic Hamiltonian equations can be approximated by a solution of the dynamical equations (2.6). Unfortunately, this fact is rigorously proved only for $d=2$. This result, formulated below, belongs to Stuart ([13]) (as also the above derivation of the adiabatic equations).

THEOREM 3. *Let $(p_\mu^0(\tau),q_\mu^0(\tau))$ be a solution of the Hamiltonian equations (2.28) with the Hamiltonian $H_{\rm ad}=T_{\rm ad}+U_{\rm ad}$, given by the formulae (2.25),(2.26), and the initial condition: $p_\mu^0(0)=p_\mu^0$, $q_\mu^0(0)=q_\mu^0$, $\mu=1,\dots,4$. Then for $\lambda=1\pm\epsilon^2$ and sufficiently small $\epsilon>0$ there exists a solution of the dynamical equations (2.6) on the interval $[0,T=O(1/\epsilon)]$ of the form*

$$\tilde A(t)=A(q(t))+id\chi+\epsilon^2a\ ,\ \tilde\Phi(t)=\Phi(q(t))e^{-i\chi}+\epsilon^2\varphi\ ,\tau=\epsilon t\ ,$$

where the pair (a,φ) satisfies the gauge condition and adiabatic equation

$$\delta^*_{(A,\Phi)}(a,\varphi)=0 \quad , \quad \langle (a,\varphi),n_\mu\rangle = 0 \ , \mu=1,\dots,4 \ ,$$

and the path $t\mapsto (q_\mu(t),p_\mu(t))$ has the form

$$q_\mu(t)=q_\mu(t)^0+\epsilon q_\mu(t)^1 \quad , \quad p_\mu(t)=p_\mu(t)^0+\epsilon p_\mu(t)^1 \ .$$

Moreover, the perturbation (a,φ) (resp. $(\dot a,\dot\varphi)$) is continuous in ϵ in the Sobolev space H_3 (resp. H_2), the gauge transform $t\mapsto\chi(t)$ is C^2-smooth in t, and the perturbations q^1_μ (resp. p^1_μ) belong to the space $C^2([0,T])$ (resp. $C^1([0,T])$).

2.5. Abrikosov strings

Euler–Lagrange equations. Abrikosov strings in the space $\mathbb{R}^3$ with coordinates (x_1,x_2,x_3) also admit an approximate description in terms of the moduli space $\mathcal{M}_d$ of solutions of vortex equations. Consider a Euclidean version of the Abelian Higgs model, described by the energy functional on $\mathbb{R}^3$, having the form

$$E(A,\Phi)=\frac{1}{2}\int\left\{|F_A|^2+|d_A\Phi|^2+\frac{\lambda}{4}(1-|\Phi|^2)^2\right\}d^3x \tag{2.29}$$

where A is a U(1)-connection on $\mathbb{R}^3$, given by the 1-form

$$A=\sum_{i=1}^3 A_i dx_i$$

with smooth pure imaginary coefficients $A_i=A_i(x_1,x_2,x_3)$, $i=1,2,3$; $\Phi=\Phi(x_1,x_2,x_3)$ is a smooth complex-valued function on $\mathbb{R}^3$; $F_A=dA$ and $d_A=d+A$ where $d=\sum_{i=1}^3\partial_i dx_i$.

The Euler–Lagrange equations for this functional have the form

$$\begin{cases} d^*F_A = i\mathrm{Im}(\bar\Phi d_A\Phi) \\ d^*_A d_A\Phi=\dfrac{\lambda}{2}\Phi(1-|\Phi|^2) \ . \end{cases} \tag{2.30}$$

Abrikosov equation. We call $\xi\mapsto[A(\xi),\Phi(\xi)]$ in $\mathcal{M}_d$ the *adiabatic path* for the functional (2.29), if it is extremal for the functional $E(A,\Phi)$, restricted to the paths, lying in $\mathcal{M}_d$. We fix the gauge, as also in the hyperbolic case, by the condition

$$\delta^*_{(A,\Phi)}(\partial_\xi A,\partial_\xi\Phi)=0 \ . \tag{2.31}$$

As in Sec.2.3 it is proved that the adiabatic paths should satisfy the adiabatic equation

$$\left(-\partial^2_\xi A,-\partial^2_\xi\Phi+\frac{1-\lambda}{2}\Phi(1-|\Phi|^2)\right)\perp T_{(A,\Phi)}\mathcal{M}_d$$

or

$$\left(-\partial^2_\xi A,-\partial^2_\xi\Phi+\frac{1-\lambda}{2}\Phi(1-|\Phi|^2)\right)\perp \mathrm{Ker}\, D_{(A,\Phi)} \ ,$$

which means in terms of the base $\{n_\mu\}$ of L^2-solutions of the equation $D_{(A,\Phi)}n_\mu = 0$ that

$$\left\langle \left(-\partial_\xi^2 A, -\partial_\xi^2 \Phi + \frac{1-\lambda}{2}\Phi(1-|\Phi|^2) \right), n_\mu \right\rangle = 0 \quad , \ \mu = 1, \ldots, 2d \ . \tag{2.32}$$

This is a Hamiltonian equation on $T^*\mathcal{M}_d$ with the Hamiltonian

$$H_{\rm ad}(A,\Phi) = \frac{1}{2}\left\{ \|\partial_\xi A\|_{L^2}^2 + \|\partial_\xi \Phi\|_{L^2}^2 + \frac{|1-\lambda|}{4}\|1-|\Phi|^2\|_{L^2}^2 \right\} \ .$$

The adiabatic equation (2.32) may be also obtained, as in Sec.2.3, by considering the perturbation

$$\tilde{A} = A(q(x_3)) + \epsilon^2 a \quad , \quad \tilde{\phi} = \Phi(q(x_3)) + \epsilon^2 \varphi$$

of the path $x_3 \mapsto [A(q(x_3)), \Phi(q(x_3))]$ in $\mathcal{M}_d$, determined, according to the Taubes' theorem, by the path $x_3 \mapsto q(x_3)$ in $S^d\mathbb{C} = \mathbb{C}^d$. Introducing the variable $\xi = \epsilon x_3$ and plugging $(\tilde{A}, \tilde{\Phi})$ into the Euler–Lagrange equations (2.30), we obtain a relation of the form

$$\langle \partial_\xi^2(A,\Phi), n_\mu\rangle + \frac{\lambda-1}{2}\langle (0, \Phi(1-|\Phi|^2), n_\mu\rangle = \epsilon h$$

where h denotes the sum of nonlinear terms, containing the currents and derivatives of n_μ with respect to ξ. The last equation for $\epsilon = 0$ coincides with the adiabatic equation

$$\left\langle \left(\partial_\xi^2 A, \partial_\xi^2 \Phi + \frac{\lambda-1}{2}\Phi(1-|\Phi|^2) \right), n_\mu \right\rangle = 0 \quad , \ \mu = 1, \ldots, 2d \ , \tag{2.33}$$

which it's natural to call in this case the *Abrikosov equation*, since its solutions describe the Abrikosov strings in $\mathbb{R}^3$, slightly differing from the direct lines, parallel to the (x_3)-axis.

Chapter III. DIMENSION FOUR: SEIBERG–WITTEN EQUATIONS

3.1. Seiberg–Witten equations on a Riemannian 4-manifold

Spin^c-structures. Let X be a connected compact oriented 4-dimensional Riemannian manifold. Assume that it is provided with a Spin^c-*structure*. It means that a complex Hermitian vector bundle $W \to X$ of rang 4 is given, which is called the *spinor bundle*, with a fiberwise linear homomorphism

$$\Gamma : TX \longrightarrow \operatorname{End} W \ ,$$

called the *Clifford multiplication*, which satisfies the relations

$$\Gamma(v)^* + \Gamma(v) = 0 \quad , \quad \Gamma(v)^*\Gamma(v) = |v|^2 \cdot \mathrm{id} \tag{3.1}$$

for any vector field $v \in \mathrm{Vect}(X)$ where * denotes the Hermitian conjugation in the space $\operatorname{End} W$.

Such a homomorphism $\Gamma : TX \to \operatorname{End} W$ extends uniquely to an isomorphism

$$\Gamma : \mathrm{Cl}^c(X) \longrightarrow \operatorname{End} W$$

where $\mathrm{Cl}^c(X) = \mathrm{Cl}^c(TX)$ is the bundle of complexified Clifford algebras on X (cf. [5]). This implies that the spinor bundle is decomposed into the direct sum

$$W = W^+ \oplus W^-$$

of its complex vector subbundles $W^\pm$ of rang 2, called the *semi-spinor bundles.* The semi-spinor bundles $W^\pm$ have the same determinant bundle

$$L_\Gamma = \det W^+ = \det W^- \ ,$$

which we shall call the *characteristic bundle* of the Spin^c-structure Γ. Accordingly, the first Chern class $c_1(L_\Gamma)$ of the line bundle L_Γ is called the *characteristic class* of the Spin^c-structure Γ.

Note that any compact oriented 4-manifold X can be provided with a Spin^c-structure, and the set of (non-equivalent) Spin^c-structures on X is parametrized by points of $H^2(X, \mathbb{Z})$ (cf. [9]).

Spin^c-connections. Consider the Levi-Civita connection ∇ on X, generated by the Riemannian metric on X. Its extension to the spinor bundle W is called a Spin^c-*connection.* In more detail, we call by a Spin^c-connection on W any connection on W (denoted by the same letter ∇), which satisfies the relation

$$\nabla_v(\Gamma(u)\Phi) = \Gamma(u)\nabla_v\Phi + \Gamma(\nabla_v u)\Phi \tag{3.2}$$

for any spinor field $\Phi \in C^\infty(X, W)$ and any vector fields $v, u \in \mathrm{Vect}(X)$. Any Spin^c-connection ∇ on W preserves the semi-spinor bundles $W^\pm$, and any two Spin^c-connections on W differ by a 1-form $\alpha \in \Omega^1(X, i\mathbb{R})$ with pure imaginary values.

Denote by

$$P_\Gamma \longrightarrow X$$

the principal Spin$^c(4)$-bundle over X, constructed using the Clifford multiplication $\Gamma : TX \to \operatorname{End} W$, so that W is the vector bundle, associated with P_Γ:

$$W = P_\Gamma \times_{\mathrm{Spin}^c(4)} \mathbb{C}^4 \ .$$

In its terms a Spin^c-connection is given by the connection 1-form $\mathcal{A} \in \Omega^1(P_\Gamma, \mathfrak{g})$ where

$$\mathfrak{g} = \mathrm{so}(4) \oplus i\mathbb{R}$$

is the Lie algebra of the group Spin$^c(4)$. Accordingly, the form $\mathcal{A}$ is decomposed into the sum

$$\mathcal{A} = \mathcal{A}_0 + A$$

of the traceless component $\mathcal{A}_0$ and the trace component $A = \frac{1}{4}\mathrm{Tr}\,\mathcal{A}$. The traceless component $\mathcal{A}_0$ induces a connection on TX, which, according to the definition of Spin^c-connection, must coincide with the Levi-Civita connection. Hence, the form $\mathcal{A}$ is completely determined by the trace component $A \in \Omega^1(P_\Gamma, i\mathbb{R})$. If the characteristic bundle L_Γ admits a square root, i.e. there exists a line bundle $L_\Gamma^{1/2} \to X$, such that

$$L_\Gamma = L_\Gamma^{1/2} \otimes L_\Gamma^{1/2} \ ,$$

then the 1-form A can be interpreted as a connection form on $L_\Gamma^{1/2}$. In general, the form $2A$ is the connection form of a connection on L_Γ, and the form A itself may be considered as a virtual connection on the virtual bundle $L_\Gamma^{1/2}$.

The Dirac operator. Denote by

$$\nabla_A : C^\infty(X, W) \longrightarrow \Omega^1(X, W)$$

the exterior covariant derivative, associated with the Spin^c-connection $\mathcal{A}$, completely determined by a form $A \in \Omega^1(P_\Gamma, i\mathbb{R})$. The *Dirac operator*, associated with the Spin^c-connection ∇_A:

$$D_A : C^\infty(X, W^+) \longrightarrow C^\infty(X, W^-) \ ,$$

is given by the local formula

$$D_A \Phi = \sum_{\nu=1}^{4} \Gamma(e_\nu) \nabla_{A, e_\nu} \Phi \qquad (3.3)$$

where $\Phi \in C^\infty(X, W^+)$, and $\{e_\nu\}$ is a local orthonormal base of TX in a neighbourhood of a given point $x \in X$ (and this definition does not depend on the choice of the base $\{e_\nu\}$).

Seiberg–Witten equations. The *Seiberg–Witten equations* or, briefly, the SW-equations are imposed on a pair (A, Φ) where $A \in \Omega^1(P_\Gamma, i\mathbb{R})$, and $\Phi \in C^\infty(X, W^+)$, and have the following form

$$\begin{cases} D_A \Phi = 0 \\ F_A^+ = \Phi \otimes \Phi^* - \dfrac{1}{2}|\Phi|^2 \cdot \mathrm{id} \end{cases} \qquad (3.4)$$

where F_A^+ is the self-dual part of the curvature form $F_A \in \Omega^2(X, i\mathbb{R})$ of the connection A. The second Seiberg–Witten equation has the following meaning. We associate with the spinor $\Phi \in C^\infty(X, W^+)$ a traceless Hermitian endomorphism W^+ by the formula

$$(\Phi \otimes \Phi^*)_0 := \Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \cdot \mathrm{id} \ .$$

The Clifford multiplication Γ assigns to a 2-form on X an endomorphism of W and, in particular, establishes an isomorphism

$$\sigma^+ : \mathrm{Herm}_0(W^+) \longrightarrow \Lambda_+^2(T^*X, i\mathbb{R})$$

between the spaces of Hermitian traceless endomorphisms of W^+ and self-dual 2-forms on X with pure imaginary values. Using this isomorphism, we can rewrite the second SW-equation in a more correct form as

$$F_A^+ = \sigma^+(\Phi \otimes \Phi^*)_0 \ . \tag{3.5}$$

As in the two-dimensional case, solutions of the Seiberg–Witten equations minimize the energy functional which has in this case the form

$$E(A, \Phi) = \int_X \left\{ |F_A|^2 + |\nabla_A \Phi|^2 + \frac{|\Phi|^2}{4}(s + |\Phi|^2) \right\} d\mathrm{vol} \tag{3.6}$$

where s is the scalar curvature of the Riemannian metric on X. For this functional there is an analogue of the Bogomolny transform, which makes it possible to rewrite it in the form

$$E(A, \Phi) = \int_X \left\{ |D_A \Phi|^2 + 2|F_A^+ - \sigma^+(\Phi \otimes \Phi^*)_0|^2 \right\} d\mathrm{vol} - -\pi^2 \langle c_1(L_\Gamma)^2, [X] \rangle$$

where

$$\langle c_1(L_\Gamma)^2, [X] \rangle = \frac{1}{\pi^2} \int_X F_A \wedge F_A \ .$$

It follows that

$$E(A, \Phi) \geq -\pi^2 \langle c_1(L_\Gamma)^2, [X] \rangle \ ,$$

and the equality is achieved precisely on solutions of the Seiberg–Witten equations.

Transformations. One of the main features of the Seiberg–Witten equations is their scale non-invariance. The change of the scale of Riemannian metric g on X:

$$g \longmapsto \tilde{g} := \lambda^2 g \quad , \quad \lambda > 0 \ ,$$

yields the following relation for the energy

$$E(A, \Phi; g) = E(A, \frac{1}{\lambda}\Phi; \tilde{g}) \ .$$

In other words, a pair (A, Φ) satisfies the Seiberg–Witten equations for the metric $g \iff$ the pair $(A, \frac{1}{\lambda}\Phi)$ satisfies the Seiberg–Witten equations for the metric $\tilde{g} = \lambda^2 g$.

There is a natural action of the group

$$\mathcal{G} := \mathrm{Map}(X, \mathrm{U}(1))$$

of *gauge transforms* on the space of pairs (A, Φ) where A is a virtual connection on the virtual bundle $L_\Gamma^{1/2}$, and $\Phi \in C^\infty(X, W^+)$, given by the formula

$$A \longmapsto A + u^{-1} du \quad , \quad \Phi \longmapsto u^{-1}\Phi \tag{3.7}$$

for $u \in \mathrm{Map}(X, \mathrm{U}(1))$. The Seiberg–Witten equations are invariant under this action. As before, we define the moduli space of solutions of SW-equations as the quotient of the space of solutions (A, Φ) of the equations (3.4) modulo gauge transforms (3.7).

The Seiberg–Witten invariant. The group $\mathcal{G}$ acts freely on the space of pairs (A, Φ) with $\Phi \neq 0$. So we can expect that the regular part of the moduli space, consisting of solutions (A, Φ) with $\Phi \neq 0$, is a smooth manifold. We can achieve the absence of solutions of the form $(A, 0)$, replacing the SW-equations (3.4) by the *modified* SW-*equations*, which are obtained from the SW-equations (3.4) by the perturbation of the form

$$\begin{cases} D_A\Phi = 0 \\ F_A^+ + \eta = \sigma^+(\Phi \otimes \Phi^*)_0 \end{cases} \tag{3.8}$$

where η is an arbitrary self-dual 2-form, $\eta \in \Omega_+^2(X, i\mathbb{R})$. If the manifold X satisfies the topological condition

$$b_2^+(X) := \dim H_2^+(X, \mathbb{Z}) > 1 \ ,$$

then one can always guarantee that for the form η in general position (i.e. belonging to a certain subset of the 2nd Baire category) the SW-equations (3.8) will not have solutions of the form $(A, 0)$ and the moduli space

$$\mathfrak{M}_\eta(X, \Gamma; g) = \frac{\{\text{solutions } (A, \Phi) \text{ of SW-equations } (3.8)\}}{\mathcal{G}}$$

will be a smooth compact oriented manifold of the even dimension (if it is not empty)

$$\frac{\langle c_1(L_\Gamma)^2, [X]\rangle - 2\chi(X) - 3\sigma(X)}{4}$$

where $\chi(X)$ is the Euler characteritic, and $\sigma(X)$ is the signature of the manifold X (cf. [9]).

So we can consider the fundamental homology class $[\mathfrak{M}_\eta(X, \Gamma; g)]$. It can be shown (cf. [9]) that for $b_2^+(X) > 1$ (we always assume in the sequel that this condition is satisfied) this homology class does not depend on the choice of the metric g and the form η in general position. Hence we can construct from this class a new smooth invariant, called the *Seiberg–Witten invariant* of the manifold X. In the most important case, when the dimension of $\mathfrak{M}_\eta(X, \Gamma; g)$ is equal to zero, i.e. the space $\mathfrak{M}_\eta(X, \Gamma; g)$ consists of a finite number of points, provided with signs (determined by the orientation of $\mathfrak{M}_\eta(X, \Gamma; g)$), this invariant is simply equal to their algebraic sum (the definition in the general case cf. in [9]). The Seiberg–Witten invariant depends only on the Spin^c-structure Γ and does not change under the diffeomorphisms of X, preserving the orientation. We shall denote this invariant by $SW(X, \Gamma)$.

3.2. Seiberg–Witten equations on a symplectic 4-manifold

Canonical Spin^c-structure on a Kähler surface. We start from the case, when X is a Kähler surface, provided with a complex structure $J \in C^\infty(X, \operatorname{End} TX)$, compatible with the Riemannian metric and orientation. Denote by ω the symplectic 2-form on X, related to the metric g and the complex structure J by the equality

$$\omega(u, v) = g(u, Jv) \tag{3.9}$$

for any $u, v \in \mathrm{Vect}(X)$. The surface X has a *canonical* Spin^c*-structure* $(W_{\mathrm{can}}, \Gamma_{\mathrm{can}})$, which is defined as follows. The canonical spinor bundle W_{can} is

$$W_{\mathrm{can}} := \Lambda^{0,*}(T^*X) = \bigoplus_{q=0}^{2} \Lambda^{0,q}(T^*X) \ . \tag{3.10}$$

The canonical spinor representation $\Gamma_{\mathrm{can}} : TX \to \mathrm{End}\, W_{\mathrm{can}}$ is given by the formula

$$\Gamma_{\mathrm{can}}(v)w := v^{0,1} \wedge w - v^{0,1} \lrcorner w \tag{3.11}$$

for $v \in \mathrm{Vect}(X)$, $w \in \Lambda^{0,q}(T^*X)$ where $v^{0,1}$ is the $(0,1)$-component of the covector $v^* \in C^\infty(X, T^*X)$, dual to a vector $v \in \mathrm{Vect}(X)$: $v^*(\cdot) := (v, \cdot)$, in the decomposition

$$v^* = v^{1,0} + v^{0,1} \ .$$

Then

$$\begin{aligned} &W^+_{\mathrm{can}} = \Lambda^{0,\cdot}(T^*X) = \Lambda^0(T^*X) \oplus \Lambda^{0,2}(T^*X) \quad , \quad W^-_{\mathrm{can}} = \Lambda^{0,1}(T^*X) \ , \\ &L_{\mathrm{can}} = \Lambda^{0,2}(T^*X) =: K^* \text{ is the anticanonical bundle } X \ . \end{aligned}$$

Any other Spin^c-structure on X is obtained from the canonical one by the tensor product with some Hermitian line bundle $E \to X$, so that the corresponding spinor and characteristic bundles have the form

$$W_E := W_{\mathrm{can}} \quad , \quad L_E := K^* \otimes E^2 \ . \tag{3.12}$$

Seiberg–Witten equations on a Kähler surface. On the canonical spinor bundle W_{can} there exists a canonical Spin^c-connection ∇_{can}, which coincides with the Levi-Civita connection on $\Omega^{0,q}(X)$. The corresponding trace component $2A_{\mathrm{can}}$ coincides with the Levi-Civita connection form on the anticanonical bundle $L_{\mathrm{can}} = K^*$. An arbitrary Spin^c-connection on the spinor bundle W_E, associated with a line bundle $E \to X$, has the following form. Fix an arbitrary Hermitian connection B on E and define a Spin^c-connection on W_E by setting

$$\nabla_A := \nabla_{A_{\mathrm{can}}+B} \ , \tag{3.13}$$

so that

$$\nabla_A(w \otimes \varphi) = (\nabla_{\mathrm{can}} w) \otimes \varphi + w \otimes (d_B \varphi)$$

for $w \in \Omega^{0,q}(X)$, $\varphi \in \Omega^0(X, E)$.

The Dirac operator D_A, constructed from the Spin^c-connection ∇_A on W_E,

$$D_A : \Lambda^0(T^*X) \oplus \Lambda^{0,2}(T^*X) \longrightarrow \Lambda^{0,1}(T^*X)$$

is given by the formula

$$D_A = D_{A_{\mathrm{can}}+B} = \sqrt{2}(\bar\partial_B + \bar\partial^*_B) \ . \tag{3.14}$$

An arbitrary section $\Phi \in C^\infty(X, W)$ can be written in the form

$$\Phi = (\varphi_0, \varphi_2) \in \Omega^0(X, E) \oplus \Omega^{0,2}(X, E) \ .$$

We write out the Seiberg–Witten equations (3.8) on the Kähler surface X for the Spin^c-structure (W_E, Γ_E) where $\Gamma_E := \Gamma \otimes \mathrm{id}$. For that we note that the bundle of self-dual forms on a Kähler surface admits the following representation

$$\Lambda^2_+ = \mathbb{R}\omega \oplus \Lambda^{2,0} \oplus \Lambda^{0,2} = \mathbb{R}\omega \oplus \overline{\Lambda^{0,2}} \oplus \Lambda^{0,2} \ .$$

According to this representation, it is convenient to write down the second SW-equation (3.8) for the self-dual curvature as two equations: an equation for the $(0,2)$-component of the curvature and an equation for its component, parallel to ω (the equation for the $(2,0)$-component of the curvature is obtained from the equation for the $(0,2)$-component by the complex conjugation). As a result we get the following system of equations

$$\begin{cases} \bar\partial_B \varphi_0 + \bar\partial_B^* \varphi_2 = 0 \\ F_B^{0,2} + \eta^{0,2} = \dfrac{\bar\varphi_0 \varphi_2}{2} \\ F^\omega_{A_{\mathrm{can}}} + F^\omega_B + \eta^\omega = \dfrac{i}{4}(|\varphi_0|^2 - |\varphi_2|^2) \end{cases} \tag{3.15}$$

where we denote by w^ω the component of a 2-form $w \in \Omega^2(X, \mathbb{C})$, parallel to ω, i.e. $w^\omega := \omega \lrcorner w$, $w^\omega = (w, \omega)$.

The analysis of the Seiberg–Witten equations on a Kähler surface. We study the SW-equations (3.15) in more detail in the case, when the form η has the type $(1,1)$. For that apply the operator $\bar\partial_B$ to the first equation (3.15). We get

$$\bar\partial_B \bar\partial_B^* \varphi_2 = -\bar\partial_B \bar\partial_B \varphi_0 = -F_B^{0,2} \varphi_0 = -\frac{1}{2}|\varphi_0|^2 \varphi_2$$

by virtue of the 2nd equation (3.15). Integrating the obtained relation over X with φ_2, we get

$$\int_X \left\{ |\bar\partial_B^* \varphi_2|^2 + \frac{1}{2}|\varphi_0|^2 |\varphi_2|^2 \right\} d\mathrm{vol} = 0 \ ,$$

whence

$$\bar\partial_B^* \varphi_2 = \bar\partial_B \varphi_0 = \bar\varphi_0 \varphi_2 \equiv 0 \ .$$

If φ_2 is not identically zero, then φ_0 vanishes on some open subset of X and by the uniqueness theorem for solutions of the equation $\bar\partial_B \varphi_0 = 0$ is identically zero.

In order to find out, which of the components φ_0, φ_2 must vanish, we integrate the 3rd equation (3.15) over X, rewriting it beforehand in the form

$$i\left(F^\omega_{A_{\mathrm{can}}} + F^\omega_B + \eta^\omega\right) = \frac{1}{4}(|\varphi_2|^2 - |\varphi_0|^2) \ .$$

Taking into account that $d\mathrm{vol} = \frac{\omega\wedge\omega}{2}$, we shall have

$$\frac{\|\varphi_2\|^2 - \|\varphi_0\|^2}{2} = \int_X \frac{|\varphi_2|^2 - |\varphi_0|^2}{4}\omega \wedge \omega = i \int_X (F_{A_{\mathrm{can}}} + F_B + \eta) \wedge \omega =$$
$$= \pi\left(-c_1(K) + 2c_1(E)\right) \cdot [\omega] + i \int_X \eta \wedge \omega \ , \quad (3.16)$$

since the form $\frac{i}{\pi}F_{A_{\mathrm{can}}}$ represents the Chern class $c_1(L_{\mathrm{can}}) = c_1(K^*) = -c_1(K)$, and the form $\frac{i}{2\pi}F_B$ represents the class $c_1(E)$ (we denote by $\|\cdot\|$ the L^2-norm on X).

If, in particular, $\eta = 0$, then

$$c_1(K)\cdot[\omega] > 2c_1(E)\cdot[\omega] \Longrightarrow \varphi_2 \equiv 0,\ \varphi_0 \not\equiv 0$$
$$c_1(K)\cdot[\omega] < 2c_1(E)\cdot[\omega] \Longrightarrow \varphi_0 \equiv 0,\ \varphi_2 \not\equiv 0\ .$$

Return to the equations (3.15) and consider them in the case of the trivial bundle $E = X \times \mathbb{C}$. We take for the perturbation η the form

$$\eta = -F^+_{A_{\mathrm{can}}} + i\lambda\omega \quad , \quad \lambda > 0\ ,$$

and try to find out, which of the components φ_0, φ_2 is identically zero in the considered case. The 3rd SW-equation (3.15), which has in this case the form

$$4i(dB)^\omega = 4\lambda + |\varphi_2|^2 - |\varphi_0|^2\ ,$$

implies, after the integration over X, that

$$4\lambda \mathrm{Vol}(X) + \|\varphi_2\|^2 = \|\varphi_0\|^2\ .$$

So the possibility $\varphi_2 \equiv 0$ is realized and the equations (3.15) take the form

$$\begin{cases} \bar\partial_B\varphi_0 = 0 \\ (dB)^{0,2} = 0 \\ 4i(dB)^\omega = 4\lambda - |\varphi_0|^2\ . \end{cases}$$

Since the bundle E is trivial, these equations admit the solution

$$B \equiv 0 \quad , \quad \varphi_0 \equiv 2\sqrt{\lambda}$$

and, as it is easy to show, any other solution is gauge equivalent to this one.

It follows, in particular, that the Seiberg–Witten invariant of the Kähler surface X (with $b_2^+(X) > 1$) with respect to the canonical Spin^c-structure is equal to

$$SW(X, \Gamma_{\mathrm{can}}) = 1\ .$$

Canonical Spin^c-connection on a symplectic 4-manifold. Let now (X,ω) be a compact 4-dimensional symplectic manifold, provided with an almost complex structure $J \in C^\infty(X, \mathrm{End}\, TX)$, related to the Riemannian metric g by the equality (3.9).

As in the Kähler case, there is a canonical Spin^c-structure $(W_{\mathrm{can}}, \Gamma_{\mathrm{can}})$ on X, defined, as in the Kähler case, by the formulae (3.10),(3.11), with

$$W_{\mathrm{can}} = \Lambda^{0,*}(T^*X) \quad , \quad L_{\mathrm{can}} = K^* .$$

However, in contrast with the Kähler case, the Levi-Civita connection ∇ no longer preserves the form spaces $\Omega^{0,q}(X)$ and, consequently, does not induce directly a connection on W_{can}. Nevertheless, starting from the Levi-Civita connection, one can construct a Hermitian connection $\tilde{\nabla}$, preserving the spaces $\Omega^{0,q}(X)$. This connection $\tilde{\nabla}$ acts on k-forms $w \in \Omega^k(X)$ by the formula

$$\tilde{\nabla}_v w = \nabla_v w + \frac{1}{2}(J\nabla_v J) \lrcorner w$$

for $v \in \mathrm{Vect}(X)$. The action of $\tilde{\nabla}$ on forms $w \in \Omega^{0,q}(X)$ amounts to the application of the Levi-Civita connection ∇ with the subsequent projection onto $\Omega^{0,q}(X)$.

Since the introduced connection $\tilde{\nabla}$ preserves the spaces $\Omega^{0,q}(X)$, it induces a connection on W_{can}. This connection is not, however, a Spin^c-connection, since it induces on TX the connection $\tilde{\nabla}$ and not the Levi-Civita connection ∇. The required Spin^c-connection ∇_{can} on W_{can} is given by the formula

$$\nabla_{\mathrm{can},v} w = \tilde{\nabla}_v w + \frac{1}{2}\Gamma_{\mathrm{can}}(J\nabla_v J) w$$

for $v \in \mathrm{Vect}(X)$, $w \in \Omega^{0,q}(X)$. In the Kähler case all three introduced connections ∇, $\tilde{\nabla}$ and ∇_{can} coincide, but for a non-integrable almost complex structure J they are, in general, different from each other.

Having the canonical Spin^c-connection ∇_{can} on W_{can}, we can carry over all further constructions from the Kähler case to the case of an arbitrary symplectic 4-manifold. Any Spin^c-structure (W_E, Γ_E) on X is given by the tensor product of the canonical one with some Hermitian line bundle $E \to X$, so that

$$W_E^+ = (\Lambda^0 \oplus \Lambda^{0,2}) \otimes E \quad , \quad W_E^- = \Lambda^{0,1} \otimes E \quad , L_E^- = K^* \otimes E^2 ,$$

and an arbitrary Spin^c-connection on $W_E = W_{\mathrm{can}} \otimes E$ has the form

$$\nabla_A = \nabla_{A_{\mathrm{can}} + B}$$

where B is some Hermitian connection on E.

Seiberg–Witten equations on a symplectic 4-manifold.

The Seiberg–Witten equations on a symplectic 4-manifold X has almost the same form, as in the Kähler case

$$\begin{cases} \bar{\partial}_B \varphi_0 + \bar{\partial}_B^* \varphi_2 = 0 \\ F_{A_{\mathrm{can}}}^{0,2} + F_B^{0,2} + \eta^{0,2} = \dfrac{\bar{\varphi}_0 \varphi_2}{2} \\ F_{A_{\mathrm{can}}}^{\omega} + F_B^{\omega} + \eta^{\omega} = \dfrac{|\varphi_2|^2 - |\varphi_0|^2}{4i} . \end{cases} \tag{3.17}$$

Consider, as in the Kähler case, the perturbation η of the form

$$\eta = -F^+_{A_{\rm can}} + \frac{i\lambda}{4}\omega$$

where λ is a sufficiently large positive number. For such η the Seiberg–Witten equations take on the form

$$\begin{cases} \bar\partial_B\varphi_0 + \bar\partial^*_B\varphi_2 = 0 \\ 2F^{0,2}_B = \bar\varphi_0\varphi_2 \\ 4iF^\omega_B = \lambda(1+|\varphi_2|^2 - |\varphi_0|^2)\ , \end{cases}$$

but, in contrast with the Kähler case,

$$\bar\partial_B\bar\partial_B\varphi_0 = F^{0,2}_B\varphi_0 - \frac{1}{4}(\partial_B\varphi_0)\circ N_J$$

where N_J is the Nijenhuis tensor of the almost complex structure J:

$$\frac{1}{2}N_J(u,v) := J(\nabla_v J)u - J(\nabla_u J)v \tag{3.18}$$

for $u, v \in \mathrm{Vect}(X)$.

We introduce the scale normalization by setting

$$\alpha := \frac{\varphi_0}{\sqrt{\lambda}} \quad , \quad \beta := \frac{\varphi_2}{\sqrt{\lambda}}\ ,$$

so that the Seiberg–Witten equations will be written in the form

$$\begin{cases} \bar\partial_B\alpha + \bar\partial^*_B\beta = 0 \\ 2F^{0,2}_B = \lambda\bar\alpha\beta \\ \dfrac{4i}{\lambda}F^\omega_B = 1+|\beta|^2 - |\alpha|^2)\ . \end{cases} \tag{3.19}$$

The analysis of the Seiberg–Witten equations on a Kähler surface. Using the relation (3.18) and Weitzenböck formula for the $\bar\partial_B$-operator

$$2\bar\partial^*_B\bar\partial_B\alpha = d^*_Bd_B\alpha - 2iF^\omega_B\alpha\ ,$$

it can be shown with the help of transformations, similar to transformations, used in the Kähler case (cf. [9]), that a solution of the equations (3.19) must satisfy the following relation (the replacement of the relation (3.16) in the Kähler case)

$$\|d_B\alpha\|^2 + \frac{\lambda}{2}\|\bar\alpha\beta\|^2 + 2\lambda\|\beta\|^2 + \frac{\lambda}{2}\int_X(4-|\alpha|^2)^2 d\mathrm{vol} =$$
$$= 8\pi^2c_1(E)\cdot[\omega] + \frac{1}{2}\langle(\partial_B\alpha)\circ N_J, \beta\rangle\ .$$

It implies that there exists a constant $C > 0$ (depending only on the almost complex structure J) such that for any $\delta > 0$ we have the following inequality

$$\left(1 - C\frac{\delta}{4}\right) \|d_B\alpha\|^2 + \frac{\lambda}{2}\|\bar{\alpha}\beta\|^2 + \left(2\lambda - \frac{1}{4\delta}\right) \|\beta\|^2 + \frac{\lambda}{2}\int_X (4 - |\alpha|^2)^2 d\mathrm{vol} \leq$$
$$\leq 8\pi^2 c_1(E) \cdot [\omega] \ . \quad (3.20)$$

Choosing the $\delta < 4/C$ and $\lambda > 1/(8\delta)$, we deduce from the last inequality a necessary condition of the solvability of the equations (3.19) for large λ

$$c_1(E) \cdot [\omega] \geq 0 \ .$$

(In the case of a Kähler surface X with $b_2^+(X) > 1$ this inequality is fulfilled automatically).

In particular, for $c_1(E) \cdot [\omega] = 0$ any solution should satisfy the relations

$$d_B\alpha = 0 \ , \ |\alpha| = 2 \ , \ \beta = 0 \ .$$

They imply that the bundle E should admit a section α without zeros, and so E is trivial. But for the trivial E it is proved, as in the Kähler case, that any solution of the equations (3.19) is gauge equivalent to the solution of the form

$$B \equiv 0 \ , \ \alpha \equiv 2 \ , \ \beta \equiv 0 \ .$$

In other words, the Seiberg–Witten invariant of the symplectic 4-manifold X (with $b_2^+(X) > 1$) with respect to the canonical Spin^c-structure is equal to

$$SW(X, \Gamma_{\mathrm{can}}) = 1 \ .$$

For an arbitrary Spin^c-structure (W_E, Γ_E), associated with a Hermitian line bundle $E \to X$, the above argument shows that the non-triviality of the Seiberg–Witten invariant, i.e. $SW(X, \Gamma_E) \neq 0$, yields the relation

$$0 \leq c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega] \ ,$$

and the equality in the left sign $\leq$ is possible only for the trivial $E = X \times \mathbb{C}$, while in the right sign $\leq$ it is realized only for $E = K$.

3.3. From Seiberg–Witten equations to pseudoholomorphic curves

The direct Taubes construction. The so called "Taubes equation": SW = Gr , mentioned in the introduction to this paper, is based on the (direct) Taubes construction, assigning to solutions of the SW-equations (3.19) in the limit $\lambda \to \infty$ pseudoholomorphic curves on X. Namely, we have the following

TAUBES THEOREM. *Let (X,ω) be a compact symplectic 4-manifold with $b_2^+(X) > 1$, provided with a compatible almost complex structure J. Let $E \to X$ be a Hermitian line bundle and (W_E, Γ_E) the corresponding Spin^c-structure on X. If $SW(X,\Gamma_E) \neq 0$, then there exists a compact pseudoholomorphic curve C, embedded into X, which homology class $[C]$ is Poincaré dual to $c_1(E)$.*

We explain how to construct such a curve C. Consider again the Seiberg–Witten equations (3.19). Since $\mathrm{SW}(X,\Gamma_E) \neq 0$, these equations necessarily have a solution $(B_\lambda,(\alpha_\lambda,\beta_\lambda))$ for any $\lambda > 0$. This solution satisfies the following apriori estimates:

$$\begin{gathered}|\alpha_\lambda| \leq 1 + \frac{C_1}{\lambda} \quad , \quad |\beta_\lambda| \leq \frac{C_2}{\lambda}(1-|\alpha_\lambda|^2) + \frac{C_3}{\lambda} \\ \|\bar\partial_B \alpha_\lambda\|^2 + \|d_B\beta_\lambda\|^2 \leq \frac{C_4}{\lambda}\end{gathered} \tag{3.21}$$

and the integral estimate

$$2\pi c_1(E)\cdot[\omega] - \frac{C}{\lambda} \leq \frac{\lambda}{2}\int_X \left(1-|\alpha_\lambda|^2\right) d\mathrm{vol} \leq 2\pi c_1(E)\cdot[\omega] + \frac{C}{\lambda}\ . \tag{3.22}$$

These estimates imply that for $\lambda \to \infty$ the section α_λ of the bundle E tends in absolute value to 1 almost everywhere on X, approaching a holomorphic section in the sense that $\|\bar\partial_B\alpha_\lambda\| \to 0$ for $\lambda \to \infty$. At the same time the section β_λ of the bundle $K^* \otimes E$ tends to zero (together with its derivatives of the 1st order) everywhere on X.

Denote by

$$C_\lambda := \alpha_\lambda^{-1}(0)$$

the zero set of the section α_λ. The required pseudoholomorphic curve C is the weak limit (in the sense of currents) of the sets C_λ.

In more detail, associate with the solution $(B_\lambda,(\alpha_\lambda,\beta_\lambda))$ a current on X of the form

$$\mathcal{F}_\lambda(\mu) = \frac{i}{2\pi}\int_X F_{B_\lambda}\wedge\mu$$

for $\mu \in \Omega^2(X)$. The currents $\{\mathcal{F}_\lambda\}$ have the uniformly bounded norm

$$\|\mathcal{F}_\lambda\| := \sup_{0\neq\mu\in\Omega^2(X)} \frac{|\mathcal{F}_\lambda(\mu)|}{\sup_{x\in X}|\mu(x)|}\ ,$$

since

$$\|\mathcal{F}_\lambda\| \leq \frac{1}{2\pi}\|F_{B_\lambda}\|_{L^1} \leq c_1(E)\cdot[\omega] + \frac{C}{\lambda}$$

according to the estimate (3.22). Therefore we can extract from the sequence $\{\mathcal{F}_\lambda\}$ its subsequence $\{\mathcal{F}_{\lambda_n}\}$, $\lambda_n \to \infty$, which is converging in the weak topology to some current $\mathcal{F}$. This current is closed and has the type $(1,1)$. Moreover, $\mathcal{F}$ is integral (i.e. it takes on integer values on forms $\mu \in H^2(X,\mathbb{Z})$) and Poincaré dual to $c_1(E)$, i.e. $\mathcal{F}(\mu) = c_1(E)\cdot\mu$ for any closed 2-form μ.

The support of the limiting current $\mathcal{F}$ coincides with the desired pseudoholomorphic curve C. The finiteness of the 2-dimensional Hausdorff measure of C follows from the estimates of the type (3.24), given below. The pseudoholomorphicity of C is proved, using the fact that C has a positive intersection index with local pseudoholomorphic discs (the proof of this fact includes the construction of a complex 1-parameter family of such discs in a neighbourhood of a given point of C).

The analysis of the Seiberg–Witten equations on $\mathbb{R}^4$. In order to study in detail the limiting behaviour of solutions $(B_\lambda, (\alpha_\lambda, \beta_\lambda))$ of the Seiberg–Witten equations in a neighbourhood of the curve C for $\lambda \to \infty$, we investigate first the reduction of these equations to their local model in a neighbourhood of a given point $x \in X$ in the scale limit $\lambda \to \infty$. We start from the local model of SW-equations, i.e. from the Seiberg–Witten equations on $\mathbb{R}^4$.

Suppose that on the space $\mathbb{R}^4$ the standard Euclidean metric g_0 and standard symplectic 2-form ω_0 are given. Thus, we fix also the standard complex structure J_0, compatible with g_0 and ω_0, with respect to which $\mathbb{R}^4$ is identified with $\mathbb{C}^2$.

The Seiberg–Witten equations (3.19) on $\mathbb{R}^4$, corresponding to $\beta = 0$ and $\lambda = 1$, have the form

$$\bar{\partial}_B \alpha = 0 \quad , \quad F_B^{0,2} = 0 \quad , \quad 4iF_B^{\omega_0} = 1 - |\alpha|^2 \tag{3.23}$$

and mean that F_B has the type $(1,1)$ with respect to J_0, and α is a holomorphic section of the trivial line bundle over $\mathbb{R}^4$, provided with the holomorphic structure, determined by $\bar{\partial}_B$.

Solutions of this system satisfy the apriori estimates of the form

$$|\alpha| \le 1 \quad , \quad |\nabla_B \alpha| \le C(1 - |\alpha|^2) \tag{3.24a}$$

$$|F_B^-| \le |F_B^+| = \frac{1}{4}(1 - |\alpha|^2) \tag{3.24b}$$

$$\int_{B_R} (1 - |\alpha|^2) d\mathrm{vol} \le CR^2 \text{ for any ball } B_R \text{ of radius } R \tag{3.24c}$$

$$\int_{\mathbb{R}^4} \left[|F_B^+|^2 - |F_B^-|^2 \right] d\mathrm{vol} \le C < \infty \ . \tag{3.24d}$$

Using these estimates, the following properties of solutions of the equations (3.23) may be proved (cf. [16]).

PROPERTY 1. *Either $|\alpha| \equiv 1$, or everywhere $|\alpha| < 1$.*

In the first case the solution (B, α) is gauge equivalent to the trivial one ($B \equiv 0, \alpha \equiv 1$). In the second case the zero set $\alpha^{-1}(0)$ is non-empty and coincides with the zero set of a complex polynomial on $\mathbb{C}^2$, whose degree is estimated via the constants in the bounds (3.24c),(3.24d).

PROPERTY 2. *Either $|F_B^-| \equiv |F_B^+|$, or everywhere $|F_B^-| < |F_B^+|$.*

In the first case there exists a $\mathbb{C}$-linear projection $\mathbb{C}^2 \to \mathbb{C}$ such that the solution (B, α) will be gauge equivalent to the pull-back of a pair (B_1, α_1), satisfying the vortex equation on $\mathbb{C}$:

$$\bar{\partial}_{B_1} \alpha_1 = 0 \quad , \quad 4iF_{B_1}^{\omega_1} = 1 - |\alpha_1|^2 \tag{3.25}$$

where ω_1 is the volume form on $\mathbb{C}$, B_1 is a Hermitian connection on the trivial line bundle over $\mathbb{C}$ with an integrable curvature F_{B_1}, α_1 is a holomorphic section of this bundle, provided with the holomorphic structure, determined by $\bar{\partial}_{B_1}$.

Property 3.

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^4}\left[|F_B^+|^2 - |F_B^-|^2\right] dvol \quad \textit{is a non-negative integer .}$$

Property 4. $1-|\alpha|^2$ *and* $|\nabla_B\alpha|$ *decay exponentially outside* $\alpha^{-1}(0)$, *i.e.*

$$(1-|\alpha|^2)+|\nabla_B\alpha|^2 \le C\exp\left\{-\frac{1}{C}dist(\cdot,\alpha^{-1}(0))\right\} \ .$$

Reduction to the local model. We return to the case of an arbitrary compact symplectic 4-manifold (X,ω), provided with a compatible almost complex structure J and, accordingly, with a compatible Riemannian metric g.

Fix an arbitrary point $x_0\in X$. Recall that the Gaussian coordinate system at x_0 determines an embedding of the space $\mathbb{R}^4$ with coordinates (y_i), $i=1,\dots,4$,

$$h:\mathbb{R}^4\hookrightarrow X\ ,$$

sending $0\in\mathbb{R}^4$ to x_0 and having the following properties:

$$h^*g = g_0 + O(|y|^2) \quad , \quad h^*\omega=\omega_0+O(|y|)\ .$$

If the triple $(B,(\alpha,\beta))$ is a solution of the SW-equations on X, then its pull-back under the embedding h is a triple of SW-data on $\mathbb{R}^4$. Applying to them the dilation

$$\delta_\lambda : y\longmapsto \frac{y}{\sqrt{\lambda}}\ ,$$

we obtain a triple of SW-data on $\mathbb{R}^4$, satisfying the SW-equations on $\mathbb{R}^4$ with $\lambda=1$. In more detail, denote by

$$\left(\underline{B},(\underline{\alpha},\underline{\beta})\right) := \delta_\lambda^* h^*\left(B,(\alpha,\beta)\right)\ .$$

Then the triple $\left(\underline{B},(\underline{\alpha},\underline{\beta})\right)$ satisfies the SW-equations on $\mathbb{R}^4$ with $\lambda=1$ for a Riemannian metric $\underline{g}$ and symplectic structure $\underline{\omega}$, slightly differing from the standard ones in the sense that

$$|\underline{g}-g_0|\le\frac{C}{r}|y|^2 \quad , \quad |\underline{\omega}-\omega_0|\le\frac{C}{\sqrt{r}}|y|$$

on the ball $B_r(0)$ of radius r with the centre at $0\in\mathbb{R}^4$.

There is the following reduction lemma, proved in [16].

Reduction lemma. *Fix constants $\epsilon>0$, $R>1$ and a natural number k. Then there exists a $\lambda_0>0$, such that for any given point $x_0\in X$ and any Gaussian coordinate system h at this point for an arbitrary solution $(B_\lambda,(\alpha_\lambda,\beta_\lambda))$ of SW-equations (3.19) on X with $\lambda\ge\lambda_0$ there exists a solution (B_0,α_0) of the model SW-equations (3.23) on $\mathbb{R}^4$, having the following properties:*

(1) $\left(\underline{B}_\lambda,(\underline{\alpha}_\lambda,\underline{\beta}_\lambda)\right) := \delta_\lambda^* h^*\left(B_\lambda,(\alpha_\lambda,\beta_\lambda)\right)$ *and* $(B_0,(\alpha_0,0))$, *maybe after a gauge transformation, will differ from each other less than ϵ in C^k-norm, referred to the ball $B_R(0)\subset\mathbb{R}^4$;*

(2) *the pair (B_0,α_0) satisfies the estimates (3.24) with a constant C, depending only on the metric g and $c_1(E)$, but not on $(B_\lambda,(\alpha_\lambda,\beta_\lambda))$.*

The limiting behaviour of solutions. Choose a sequence $\lambda_n \to \infty$ and consider the corresponding sequence of SW-solutions $(B_n, (\alpha_n, \beta_n)) := (B_{\lambda_n}, (\alpha_{\lambda_n}, \beta_{\lambda_n}))$. Using the reduction to the local model, we can study the limiting behaviour of solutions $(B_n, (\alpha_n, \beta_n))$ in a neighbourhood of an arbitrary point $x_0 \in C$. Namely, by the reduction lemma we can choose a subsequence $\{(B_n, (\alpha_n, \beta_n))\}$ (numerated, to simplify the notations, by the same subindex n), having the following property. The corresponding sequence of SW-solutions $\left(\underline{B}_n, (\underline{\alpha}_n, \underline{\beta}_n)\right)$ on $\mathbb{R}^4$ (possibly after gauge transformations) will converge in the C^∞-topology on compacts of $\mathbb{R}^4$ to a solution $(B_0, \alpha_0) \equiv (B_0, (\alpha_0, 0))$ of the model SW-equations (3.23) on $\mathbb{R}^4$.

Denote by C_0 the curve $\alpha_0^{-1}(0)$ in $\mathbb{R}^4 = \mathbb{C}^2$. Then the Gaussian embedding $h : \mathbb{R}^4 \hookrightarrow X$, sending $0 \in \mathbb{R}^4$ to $x_0 \in C \subset X$, induces an isomorphism

$$h_* : T_0\mathbb{R}^4 \cong \mathbb{R}^4 \xrightarrow{\approx} T_{x_0}X \ ,$$

so that

$$h_* : T_0C_0 \xrightarrow{\approx} T_{x_0}C \quad , \quad N_0C_0 = T_0\mathbb{R}^4/T_0C_0 \xrightarrow{\approx} N_{x_0}C$$

where $N_{x_0}C$ is the normal plane to C at the point x_0.

The Seiberg–Witten equations for (B_0, α_0) have the form

$$\bar{\partial}_{B_0}\alpha_0 = 0 \quad , \quad F_{B_0}^{0,2} = 0 \quad , \quad 4iF_{B_0}^{\omega_0} = 1 - |\alpha_0|^2 \ , \tag{3.26}$$

meaning that the form F_{B_0} have the type $(1,1)$ with respect to J_0, and also $F_{B_0}^+ = F_{B_0}^{\omega_0}$. But the form $d_{B_0}\bar{\alpha}_0 \wedge d_{B_0}\alpha_0$, due to the $\bar{\partial}_{B_0}$-holomorphicity of α_0, also has the type $(1,1)$, therefore

$$\text{the form } P_+(d_{B_0}\bar{\alpha}_0 \wedge d_{B_0}\alpha_0) \text{ is proportional to } F_{B_0}^+$$

where we denote by P_+ the orthogonal projection $P_+ : \Lambda^2 \to \Lambda^2_+$ of the space of 2-forms to the subspace of self-dual 2-forms. It follows that the restriction of $F_{B_0}^+$ to T_0C_0 is equal to zero, moreover, the value of $F_{B_0}^+$ on any pair of vectors from $T_0\mathbb{R}^4$, at least one of which belongs to T_0C_0, is equal to zero too. Thus, the Seiberg–Witten equations at the point $0 \in C_0$ reduce to equations on N_0C_0, coinciding with the vortex equations (3.25):

$$\bar{\partial}_{B_1}\alpha_1 = 0 \quad , \quad 4iF_{B_1}^{\omega_1} = 1 - |\alpha_1|^2$$

where we denote by $(B_1, \alpha_1 = \alpha_0)$ the restriction of (B_0, α_0) to N_0C_0, and by ω_1 the volume form on N_0C_0. In terms of the manifold X it means that in the indicated scale limit for $n \to \infty$ solutions $\{(B_n, (\alpha_n, \beta_n))\}$ of SW-equations (3.19) converge at the point $x_0 \in C$ to some solution of the vortex equation on the normal plane $N_{x_0}C$. Since $x_0 \in C$ was an arbitrary point, it implies that in the scale limit $\lambda \to \infty$ the solution $(B_\lambda, (\alpha_\lambda, \beta_\lambda))$ of the original SW-equations reduces on the curve C to a family of solutions of the vortex equations on normal planes N_xC, parametrized by points $x \in C$.

3.4. From pseudoholomorphic curves to the Seiberg–Witten equations

Local geometry in a neighbourhood of a pseudoholomorphic curve. Let C be a compact pseudoholomorphic curve on a symplectic 4-manifold (X,ω) and $\pi : N \to C$ is its normal bundle in X, so that

$$T_xX = T_xC \oplus N_x$$

for $x \in C$. The space N_x is identified with the orthogonal complement to T_xC in T_xX, so the complex structure operator $J_x : T_xX \to T_xX$, which is an orthogonal transform, preserving T_xC, preserves N_x also. Thus, J induces an almost complex structure on N and $N \to C$ may be considered as a complex line bundle over C. On the other hand, we can introduce a fiberwise constant almost complex structure J_0 on $\pi : N \to C$, taking the pull-back

$$J_0 := \pi^*(J|C)$$

under the projection π.

The Riemannian metric g on X induces a Riemannian metric on N, generating the Levi-Civita connection θ on N. Denote by d_θ the exterior covariant derivative operator, associated with this connection. Its $(0,1)$-component with respect to J_0 determines a $\bar\partial$-operator

$$\bar\partial := \bar\partial_{J_0}$$

on N and so the structure of a holomorphic line bundle on $N \to C$. The pull-back of the connection θ under the map $\pi : N \to C$ defines a connection on $\pi^*N \to N$, denoted by the same letter θ. We have

$$TN \cong \pi^*(TC) \oplus \pi^*N \ .$$

We introduce a fiberwise J_0-holomorphic coordinate s on $\pi : N \to C$, so that $d_\theta s$ is a J_0-holomorphic 1-form on fibers of $\pi : N \to C$.

On the other hand, we can choose in $\pi : N \to C$ a disc subbundle $U \subset N$, formed by the discs of sufficiently small radius in fibers of N (for simplicity, we assume that this small radius is equal to 1), which can be identified with the help of the exponential map with a tube neighbourhood $\exp U$ of the curve C in X. The subspace $T^{1,0}_J U$ of tangent vectors in TU of the type $(1,0)$ with respect to the almost complex structure J is generated locally by an orthonormal frame, formed by the pair of 1-forms $\{\kappa_0, \kappa_1\}$ where:

(1) κ_0 is a section of $\pi^*(T^*C)$, which generates $T^{1,0}C$ at points of C;
(2) $\kappa_1 = d_\theta s + \sigma + O(|s|^2)$ where σ is a 1-form on $\pi^*(TC)$ with values in π^*N, which vanishes at points of C.

The form σ with respect to the decomposition

$$TU = T^{1,0}_{J_0}U \oplus T^{0,1}_{J_0}U =: T^{1,0}U \oplus T^{0,1}U$$

is represented in the form $\sigma = \sigma^{1,0} + \sigma^{0,1}$ where

$$\sigma^{1,0} = -\bar\nu s + \gamma\bar s + O(|s|^2) \quad , \quad \sigma^{0,1} = \nu s + \mu\bar s + O(|s|^2) \ , \tag{3.27}$$

and ν may be considered as a $(0,1)$-form on C, μ — as a $(0,1)$-form on C with values in N^2, and γ — as a $(1,0)$-form on C with values in N^2. The symplectic form ω in these terms is represented in the form

$$\omega = \frac{i}{2}(\kappa_0 \wedge \bar\kappa_0 + \kappa_1 \wedge \bar\kappa_1) \ .$$

The vortex bundle. We construct the vortex bundle over C, whose sections will coincide with families of solutions of vortex equations on the normal bundle $N \to C$.

Let, as in Ch.I, $\mathcal{M}_d$ denotes the moduli space of vortices with the vortex number d, i.e. the quotient of the space of solutions of the vortex equations (1.12) on the complex plane $\mathbb{C}$

$$\bar{\partial}_A \Phi = 0 \quad , \quad 2i(dA) = *(1 - |\Phi|^2) \quad , \; (A, \Phi) \in \Omega^1_s \times H_s \; , \tag{3.28}$$

with finite energy, modulo gauge transforms. The space $\mathcal{M}_d$, according to Ch.I, may be identified with $\mathbb{C}^d$ via the map

$$S : \mathbb{C}^d \longrightarrow \Omega^1_s \times H_s \; , \; y = (y_1, \ldots, y_d) \longmapsto (A, \Phi) = \big(i(\bar{\partial} - \partial)u, p_y e^{-u}\big) \tag{3.29}$$

where $p_y(z)$ is the polynomial in $z \in \mathbb{C}$ of degree d with coefficients $y_1, \ldots, y_d$:

$$p_y(z) = z^d + y_1 z^{d-1} + \cdots + y_d \; ,$$

and u is a solution of the Liouville type equation of the form

$$4i\partial\bar{\partial}u = * \left(1 - |p_y|^2 e^{-2u}\right) \; ,$$

subject to the asymptotic condition: $u(z) \sim d \ln|z|$ for $|z| \to \infty$.

Introduce on $\mathbb{C}^d$ a homogeneous action of the circle U(1), consistent with the above identification $S : \mathbb{C}^d \to \mathcal{M}_d$. Namely, set

$$\mathrm{U}(1) \ni e^{i\theta} : (y_1, \ldots, y_d) \longmapsto (e^{i\theta} y_1, \ldots, e^{id\theta} y_d) \; .$$

This U(1)-action on $\mathbb{C}^d$ generates the action of U(1) on $\mathcal{M}_d$, given by the map

$$\delta^*_\theta : (A, \Phi) \longmapsto (\delta^*_\theta A, e^{id\theta} \delta^*_\theta \Phi)$$

where

$$\delta_\theta : z \longmapsto e^{-i\theta} z$$

is the rotation of the complex plane $\mathbb{C}$, and we denote by $\delta^*_\theta A$, $\delta^*_\theta \Phi$ its action on A, Φ.

Along with this action we introduce also the scale transform, given by the contraction

$$\rho_r : z \longmapsto \sqrt{r} z$$

of the complex plane $\mathbb{C}$ to the scale $\sqrt{r}$ where $r > 0$. This transform, as it was mentioned before, translates solutions (A, Φ) of the vortex equations (3.28) into solutions of the equations of the form

$$\bar{\partial}_{A_r} \Phi_r = 0 \quad , \quad 2i(dA_r) = *r(1 - |\Phi_r|^2)$$

where $(A_r, \Phi_r) := (\rho^*_r A, \rho^*_r \Phi)$.

The gauge of solutions of the vortex equation may be fixed by the condition

$$\Phi(z) = z^d f(z) \quad \text{for} \quad |z| > r_0(y)$$

where $f > 0$, and $r_0(y)$ is a sufficiently large number (it may be defined by an explicit function of y, cf. Ch.I). For the *radial solution*

$$A_d = u_d(zd\bar{z} - \bar{z}dz) \quad , \quad \Phi_d = z^d f_d \tag{3.30}$$

where u_d, f_d are real-valued functions of $|z|^2$, and also $f_d > 0$.

We return to the pseudoholomorphic curve C and denote by $L \to C$ the subbundle of the unit circles in $N \to C$ with the natural $\mathrm{U}(1)$-action. Define the *vortex bundle* $\mathcal{L}_d \to C$, associated with $N \to C$, as the skew product

$$\mathcal{L}_d := L \times_{\mathrm{U}(1)} \mathcal{M}_d \longrightarrow C$$

with respect to the $\mathrm{U}(1)$-action on $\mathcal{M}_d$, introduced above. It is isomorphic to the bundle $\bigoplus_{m=1}^d N^m$ over C.

Sections of $\mathcal{L}_d$ are identified with families $\{\tau_x\}_{x\in C}$ of solutions $\tau_x = [A_x, \Phi_x]$ of the vortex equations on $\mathbb{C}$ (defined up to gauge equivalence), parametrized by points $x \in C$. Choosing in every N^m the zero section, we obtain the constant *radial section* of $\mathcal{L}_d$, assigning to every $x \in C$ the radial solution (3.30) from $\mathcal{M}_d$.

There is a natural action of the scale transforms ρ_r^* on sections of $\mathcal{L}_d$. In particular, on the radial section $\tau_d = [A_d, \Phi_d]$ it is given by the formulae

$$\begin{aligned}
&\rho_r^* : z^d f_d(|z|^2) \longmapsto r^{d/2} z^d f_d(r|z|^2) \\
&\rho_r^* : u_d(|z|^2)(zd\bar{z} - \bar{z}dz) \longmapsto r u_d(r|z|^2)(zd\bar{z} - \bar{z}dz) \ .
\end{aligned}$$

Construction of SW-data from a section of the vortex bundle. We can construct from any section τ of the vortex bundle $\mathcal{L}_d$ in a standard way a collection of data for the SW-equations on X. Let, as before, $U \to C$ denotes the disc subbundle of $N \to C$, identified with a tube neighbourhood of the curve C in X. Denote by U_δ the subbundle of discs of radius $\delta < 1$ in $U \to C$.

We construct first a complex line bundle $E \to X$, whose 1st Chern class is Poincaré dual to the homology class $d[C]$ where $[C]$ is the homology class of the curve C. Here and later on we shall assume that C is a connected pseudoholomorphic curve. In other words, instead of an arbitrary pseudoholomorphic chain $C = \sum d_k C_k$, mentioned in the introduction to this paper, we restrict ourselves to the case when this chain contains only one component. In the general case all subsequent constructions are carried out separately for each component C_k.

The bundle $E \to X$ is constructed by the smooth gluing of the trivial line bundle over $X - U_{1/2}$ and the bundle $(\pi^*N)^d$ over U where the gluing is given by the formula

$$(U - U_{1/2}) \times \mathbb{C} \ni (x, \zeta) \longleftrightarrow \left(x, \zeta\frac{\Phi_r}{|\Phi_r|}\right) \in (\pi^*N)^d \ , \tag{3.31}$$

and the number r is chosen in such a way that all zeros of Φ_r are contained in $U_{1/2}$.

Analogously, a section α_r of the bundle E is constructed with the help of the smooth gluing of the section Φ_r over a small neighbourhood of the curve C in X with the trivial section $\alpha_r \equiv 1$ outside $U_{1/2}$. In more detail, pick a $\delta < 1/2$ and denote by χ_δ a smooth cutt-off function on U, equal to 1 on U_δ and 0 outside $U_{1/2}$. Set

$$\alpha_r = \begin{cases} 1 & \text{over } X - U_{1/2} \\ \frac{\Phi_r}{\chi_\delta + (1-\chi_\delta)|\Phi_r|} & \text{over } U \ . \end{cases} \tag{3.32}$$

In order to construct a connection B_r on E, we choose a connection B_r^0 on $E|X - U_{1/2}$, for which the section α_r is covariantly constant, and glue it smoothly with the connection, equal to the sum of the connection A_r and the connection θ_d on $(\pi^* N)^d$, induced by the Levi-Civita connection θ on $\pi^* N$. In other words,

$$B_r = \begin{cases} B_r^0 & \text{over } X - U_{1/2} \\ \theta_d + \chi_\delta A_r - (1-\chi_\delta)\alpha_r^{-1}\nabla_{\theta_d}\alpha_r & \text{over } U \ . \end{cases} \tag{3.33}$$

The constructed bundle E and the pair (B_r, α_r) form a collection of SW-data (with $\beta_r \equiv 0$) for the SW-equations on X. Substituting them into the SW-equations, we obtain the discrepancy of the form

$$|D_{B_r}(\alpha_r, 0)| \le C e^{-c\sqrt{r}\,\mathrm{dist}} \tag{3.34a}$$

$$|F_{B_r}^+ + \frac{ir}{4}(1 - |\alpha_r|^2)\omega| \le C\sqrt{r} e^{-c\sqrt{r}\,\mathrm{dist}} \tag{3.34b}$$

where $r \gg 1$, and $\mathrm{dist}(\cdot) := \mathrm{dist}(\cdot, C)$ is the distance from the curve C. One can see from the estimates (3.34) that, even for $r \to \infty$, it is impossible to guarantee that the constructed SW-data will satisfy the SW-equations with any prescribed accuracy. However, one cannot call this conclusion unexpected. By analogy with the 3-dimensional case it is justified to expect that not any section τ of the vortex bundle $\mathcal{L}_d$ will give the SW-data, satisfying approximately to the SW-equations. For that this section should be a solution of the adiabatic equation. In other words, it is possible to strengthen the estimates (3.34) only in the case when τ satisfies the adiabatic equation to be derived next.

The derivation of the adiabatic equation. The derivation of the adiabatic equation is carried out along the same lines, as in the (2+1)-dimensional case in Ch.II. A new feature is the non-linearity of the operator $\bar\partial_J$. We write it in the form

$$\bar\partial_J = P_J^{0,1} \circ d = P_{J_0}^{0,1} \circ d + Q_J^{0,1} \circ d = \bar\partial_{J_0} + Q_J^{0,1} \circ d = \bar\partial_{J_0} + q_J^{0,1} \circ d + \dots \tag{3.35}$$

where we denote by $P_J^{0,1}$ the orthogonal projection onto the subspace of 2-forms of the type $(0,1)$ with respect to the complex structure J (the operator $P_{J_0}^{0,1}$ is defined analogously), and $q_J^{0,1}$ denotes the linearization of the operator $Q_J^{0,1}$ in a neighbourhood of C, so that, according to (3.27):

$$q_J^{0,1} \circ d = (\nu s + \mu \bar s) \circ d \ .$$

In analogous way,

$$d_J^+ = P_J^+ \circ d = P_{J_0}^+ \circ d + Q_J^+ \circ d = d_{J_0}^+ \circ d + Q_J^+ \circ d = d_{J_0}^+ \circ d + q_J^+ \circ d + \dots \quad (3.36)$$

where we denote by q_J^+ the linearization of the operator Q_J^+ in a neighbourhood of C.

The Seiberg–Witten equations for a pair $(B, \alpha) \equiv (B, (\alpha, 0))$ have the form

$$\begin{cases} \bar{\partial}_{J,B}\alpha = 0 \\ P_J^+(dB) + \dfrac{i}{4}(1 - |\alpha|^2)\omega = 0 \ . \end{cases}$$

If the pair $(B, \alpha) := (B_r, \alpha_r)$ is constructed from a vortex section $\tau = [A, \Phi]$, then, according to the formulae (3.32),(3.33), we shall have in the vicinity of C

$$\alpha_r = \Phi_r \quad , \quad B_r = \theta_d + A_r$$

where $(A_r, \Phi_r) := \rho_r^*(A, \Phi)$. The Seiberg–Witten equations in the vicinity of C will have the form

$$\begin{cases} \bar{\partial}_{J,B_r}\Phi_r = 0 \\ P_J^+(dB_r) + \dfrac{ir}{4}(1 - |\Phi_r|^2)\omega = 0 \end{cases}$$

or, taking into account the representations (3.35),(3.36) for the operators $\bar{\partial}_J$ and d_J^+:

$$\begin{cases} \bar{\partial}_{B_r}\Phi_r + Q^{0,1}d_{B_r}\Phi_r = 0 \\ P^+(dB_r) + \dfrac{ir}{4}(1 - |\Phi_r|^2)\omega + Q^+(dB_r) = 0 \end{cases} \quad (3.37)$$

where we omit here and later on the subindices J_0 and J of the operators.

As in the (2+1)-dimensional case, we consider the perturbation of the vortex section $[A, \Phi]$ of the form

$$\tilde{A} = A + \epsilon^2 a \quad , \quad \tilde{\Phi} = \Phi + \epsilon^2 \varphi$$

where $\epsilon = 1/\sqrt{r}$, and the pair (a, φ) satisfies the condition

$$(a, \varphi) \perp \ \mathrm{Ker}\, \mathcal{D}_{(A,\Phi)}$$

in the notations from Sec.2.2 or, in terms of the base of L^2-solutions $\{n_\mu\}$ of the equation $\mathcal{D}_{(A,\Phi)}n_\mu = 0$ from Sec.2.3:

$$\langle (a, \varphi), n_\mu \rangle = 0 \ , \ \mu = 1, \dots, 2d \ . \quad (3.38)$$

Consider the decomposition of the operator $\bar{\partial}$ into its horizontal and vertical components

$$\bar{\partial} = \bar{\partial}^C + \bar{\partial}^V$$

where $\bar\partial^C$ is the operator of the $\bar\partial$-differentiation along C, and $\bar\partial^V$ is the operator of the $\bar\partial$-differentiation along fibers of $N \to C$. The operator $\bar\partial_B$ admits the analogous representation $\bar\partial_B = \bar\partial_B^C + \bar\partial_B^V$.

We introduce now, by analogy with the "slow" time from Ch.II, new local coordinates in a neighbourhood of a considered point $x_0 \in C$ of the form: $\xi = \epsilon x$, and substitute the pair $(\tilde A, \tilde\Phi)$ into the Seiberg–Witten equations (3.37). As a result we obtain the relation of the form:

$$(\bar\partial^C_{A_r}\varphi_r, P_+(d^C a_r)) + q\mathcal{D}^{*V}_{(A_r,\Phi_r)}(a_r,\varphi_r) = -q(\partial_{A_r}\Phi_r, dA_r) + (\varphi^1_r, a^1_r) + \dots \quad (3.39)$$

where $q(\Psi, F) := (q^{0,1}\Psi, q^+F)$. We denote by (φ^1, a^1) the terms, having the zeroth order in ϵ, and by dots the terms, having higher order in ϵ. Taking the inner product of the both sides of (3.39) with n_μ and using the fact that $\mathcal{D}_{(A,\Phi)}n_\mu = 0$, we obtain, as in Sec.2.3, the adiabatic equation for the section $[A, \Phi]$ of the form

$$\left\langle \left(-(\nu s + \mu\bar s)\frac{\partial_A\Phi}{\partial s} + \varphi^1_{0,1}, -(\nu s + \mu\bar s)d_\theta\bar s + a^1_{0,2} \right), n_\mu \right\rangle = 0 \ , \ \mu = 1, \dots, 2d \ , \quad (3.40)$$

where $\frac{\partial_A\Phi}{\partial s}$ is defined by the equality $\partial_A\Phi = \frac{\partial_A\Phi}{\partial s}d_\theta s$. For the derivation of (3.40) we have used the formula (3.27) for the expressions $q^{0,1}(\partial_{A_r}\Phi_r)$ and $q^+(dA_r)$, and denoted by $\varphi^1_{0,1}$ (resp. $a^1_{0,2}$) the $(0,1)$-component of the 1-form φ^1 (resp. $(0,2)$-component of the 2-form a^1).

We add some comments on the meaning of the forms φ^1, a^1, which have arisen in the formulae (3.39),(3.40). They include the terms of the 0th order in ϵ, which depend on the variation of the section $\tau = [A, \Phi]$ in the moduli space $\mathcal{M}_d$, so they vanish for the radial section τ_d. Consider the vortex bundle $\mathcal{L}_d \to C$ and denote by Vert the vertical subbundle in $T\mathcal{L}_d$. This is a bundle of rank $2d$, whose fibre at $[A, \Phi]$ coincides with $T_{(A,\Phi)}\mathcal{M}_d$. The connection θ on N (and, consequently, on L) determines a connection on $\mathcal{L}_d$ with values in Vert. The form of this connection is a 1-form $\theta_{\mathcal{M}} := \theta_{\mathcal{M}_d}$ on $\mathcal{L}_d$ with values in Vert. Taking this into account, we can identify the pair (φ^1, a^1) with the covariant derivative of the section $\tau = [A, \Phi]$ with respect to this connection:

$$\nabla_{\theta_{\mathcal{M}}}\tau = \nabla_{\theta_{\mathcal{M}}}[A, \Phi] = (a^1, \varphi^1) \ .$$

By analogy with Sec.2.5 it is natural to call (3.40) the *complex Abrikosov equation*, and the sections $\tau = [A, \Phi]$ of the vortex bundle $\mathcal{L}_d \to C$, satisfying this equation, the *complex Abrikosov strings*. The point x, running through the curve C, plays the role of a complex parameter on the string.

The spaces of complex Abrikosov strings. We introduce the *space of complex Abrikosov strings* $\mathcal{Z}$, consisting of the sections $\tau = [A, \Phi] \in C^\infty(C, \mathcal{L}_d)$, satisfying the Abrikosov equation (3.40). Its regular part $\mathcal{Z}_{\mathrm{reg}} \subset \mathcal{Z}$ consists of the sections of $\mathcal{L}_d \to C$, for which this equation is fulfilled transversally (more explicit description of $\mathcal{Z}_{\mathrm{reg}}$ is given in [17]). The introduced spaces have the following properties (cf. [17]).

Property 1. *The space $\mathcal{Z}$ is locally compact, and its regular part $\mathcal{Z}_{\mathrm{reg}}$ is a smooth oriented manifold of dimension*

$$2d(1-g(C))+d(d+1)n(C)$$

where $g(C)$ is the genus of the curve C, and $n(C)$ is the degree of the normal bundle $N \to C$.

Property 2. *With the help of the map $S : \mathbb{C}^d \to \mathcal{M}_d$, $y \mapsto \tau = S(y)$, from the formula (3.29) the space $\mathcal{Z}$ may be identified with the space of smooth sections of the bundle*

$$\bigoplus_{m=1}^{d} N^m \longrightarrow C \ , \tag{3.41}$$

satisfying the equation

$$\bar{\partial} y + \nu\chi y + \mu F(y) = 0 \tag{3.42}$$

where $y = (y_1, \ldots, y_d)$ is a section of the bundle (3.41), χ is the map of the form

$$\chi : (y_1, \ldots, y_d) \longmapsto (y_1, 2y_2, \ldots, dy_d) \ ,$$

and F is a smooth fiberwise map

$$F : C^\infty(C, \bigoplus_{m=1}^{d} N^m) \longrightarrow C^\infty(C, \bigoplus_{m=1}^{d} N^{m-2}) \ .$$

This property shows that the complex Abrikosov equation (3.40) is, in fact, a non-linear $\bar{\partial}$-equation for sections of the bundle $\bigoplus_{m=1}^{d} N^m$.

Property 3. *For $d = 1$ the space $\mathcal{Z}$ coincides with the kernel of the operator*

$$\bar{\partial} f_\tau + \nu f_\tau + \mu \bar{f}_\tau = 0 \tag{3.43}$$

where $\tau = [A, \Phi] \in \mathcal{Z}$, and f_τ is the section of the normal bundle $N \to C$, the image of which coincides with $\Phi^{-1}(0)$. For $d > 1$, but $\mu = 0$, the space $\mathcal{Z}$ consists of the sections $y = (y_1, \ldots, y_d)$ of the bundle $\bigoplus_{m=1}^{d} N^m \longrightarrow C$, satisfying the equations

$$\bar{\partial} y_m + m\nu y_m = 0 \ , \ m = 1, \ldots, d \ .$$

The construction of an approximate solution of the Seiberg–Witten equations. We show now how to construct an arbitrarily precise approximate solution of the SW-equations from a solution of the complex Abrikosov equation (3.40).

Let $\tau = [A, \Phi]$ be a section of the vortex bundle $\mathcal{L}_d \to C$, satisfying the equation (3.40), which we rewrite in the form

$$\langle p(A, \Phi), n_\mu \rangle = 0 \ , \ \mu = 1, \ldots, 2d$$

where

$$p(\tau) = p(A, \Phi) = \left(-(\nu s + \mu \bar{s}) \frac{\partial_A \Phi}{\partial s} + \varphi^1_{0,1}, -(\nu s + \mu \bar{s}) d_\theta \bar{s} + a^1_{0,2} \right) .$$

Denote by (b, ψ) an L^2-solution of the equation

$$\mathcal{D}^*_{(A,\Phi)}(b, \psi) = p(A, \Phi) . \tag{3.44}$$

This solution is unique since $p(A, \Phi) \perp \operatorname{Ker} \mathcal{D}_{(A,\Phi)}$.

We construct new SW-data $(\tilde{B}, (\tilde{\alpha}, \tilde{\beta}))$ on the bundle E from the SW-data (B, α), given by the formulae (3.32),(3.33). We set

$$(\tilde{B}_r, (\tilde{\alpha}_r, \tilde{\beta}_r)) = \begin{cases} (B_r, (\alpha_r, 0)) & \text{on } X - U_{1/2} \\ (B_r + \chi_\delta b_r, (\alpha_r, \chi_\delta \psi_r)) & \text{on } U \end{cases} \tag{3.45}$$

where $(b_r, \psi_r) := \rho_r^*(b, \psi)$.

The substitution of the new SW-data $(\tilde{B}_r, (\tilde{\alpha}_r, \tilde{\beta}_r))$ (we omit the tilde in their notation later on) into the SW-equations yields the following discrepancy (cp. with the formula (3.34)):

$$|D_{B_r}(\alpha_r, \beta_r)| \le \frac{C}{\sqrt{r}} e^{-c\sqrt{r}\mathrm{dist}} ,$$

$$\left| F^+_{B_r} + \frac{ir}{4}(1 - |\alpha_r|^2 + |\beta_r|^2)\omega - \frac{r}{2}(\alpha_r \bar{\beta}_r - \bar{\alpha}_r \beta_r) \right| \le C e^{-c\sqrt{r}\mathrm{dist}} .$$

With the help of these estimates, in [17] the following result was proved .

THEOREM 4. *Fix a relatively compact open subset K in $\mathcal{Z}_{\mathrm{reg}}$ and denote by $\mathfrak{M}^r$ the moduli space of solutions of the SW-equations*

$$\begin{cases} D_{B_r}(\alpha_r, \beta_r) = 0 \\ F^+_{B_r} + \frac{ir}{4}(1 - |\alpha_r|^2 + |\beta_r|^2)\omega = \frac{r}{2}(\alpha_r \bar{\beta}_r - \bar{\alpha}_r \beta_r) \end{cases} \tag{3.46}$$

for $r \gg 1$ with respect to the Spin^c-structure on X, associated with the bundle $E \to X$, defined by the smooth gluing (3.31). There exists a continuous map

$$\Psi_r : K \longrightarrow \mathfrak{M}^r ,$$

assigning to a vortex section $\tau = [A, \Phi] \in K \subset \mathcal{Z}_{\mathrm{reg}}$ a solution $\Psi_r(\tau) = (\mathbf{B}_r, (\boldsymbol{\alpha}_r, \boldsymbol{\beta}_r))$ of the SW-equations (3.46) of the form

$$\mathbf{B}_r = B_r + \sqrt{r} B' , \ \boldsymbol{\alpha}_r = \alpha_r + \alpha' , \ \boldsymbol{\beta}_r = \beta_r + \beta'$$

where $(B_r, (\alpha_r, \beta_r))$ are the SW-data, constructed from the section τ by the formula (3.45), and $\gamma' := (B', (\alpha', \beta'))$ is a correction term, satisfying the estimates

$$\|\nabla \gamma'\| + \sqrt{r}\|\gamma'\| \le \frac{C}{\sqrt{r}}$$

$$\sup_X |\nabla \gamma'| + \sqrt{r} \sup_X |\gamma'| \le C .$$

In [17] an extension of this theorem to the case of a section $\tau \in \mathcal{Z} \setminus \mathcal{Z}_{\mathrm{reg}}$ is also proved.

Bibliographical notes.

The basics of the superconductivity theory and the derivation of the Ginzburg–Landau equations may be found in [6]. The vortex equations on the complex plane and the proofs of the Taubes theorems from Sec.1.3 are given in [2]. Note that in [2] a direct proof of the Theorem 1 is given, which does not employ the Kazdan–Warner theorem.

The idea of using the moduli spaces of static vortex solutions for the study of the scattering of slowly moving vortices belongs to Manton (cf. [7]). Using this idea, in [12] a description of the scattering of slowly moving vortices in the Abelian (2+1)-dimensional Higgs model was given, and in [14] a periodic solution of the type "vortex-vortex" on the Riemannian sphere was constructed. A mathematically rigorous justification of the Manton's idea in terms of the adiabatic limit was given by Stuart in [13].

In a short description of the general theory of the Seiberg–Witten equations, given in the beginning of Ch.III (Sec.3.1,3.2), we follow the book [9]. In the last two Sections of this Chapter we present the Taubes results from [16,17]. The main difference between our exposition and that of [17] is that the equation for sections of the vortex bundle, necessary for the construction of approximate solutions of the SW-equations, was postulated in [17] on formal grounds, and we obtain it, by analogy with the (2+1)-dimensional case, as a complex analogue of the adiabatic limit equation.

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