Deformation theory: A powerful tool in physics modeling*

DANIEL STERNHEIMER
Laboratoire Gevrey de Mathématique Physique, CNRS UMR 5029
Institut de Mathématiques de Bourgogne, Université de Bourgogne
BP 47870, F-21078 Dijon Cedex France.
e-mail: `Daniel.Sternheimer@u-bourgogne.fr`

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Abstract

Deformation theory has proved to be a powerful tool – so far, a posteriori – in modeling physical reality. We start with a short historical and philosophical review of the context and concentrate this rapid presentation on a few interrelated directions where deformation theory is essential in bringing a new framework – which has then to be developed using adapted tools, some of which come from the deformation aspect. Minkowskian space-time can be deformed into Anti de Sitter, where massless particles become composite (also dynamically): this opens new perspectives in particle physics, at least at the electroweak level, including prediction of new mesons. Nonlinear group representations and covariant field equations, coming from interactions, can be viewed as some deformation of their linear (free) part: recognizing this fact can provide a good framework for treating problems in this area. Last but not least, (algebras associated with) classical mechanics (and field theory) on a Poisson phase space can be deformed to (algebras associated with) quantum mechanics (and quantum field theory). That is now a frontier domain in mathematics and theoretical physics called deformation quantization, with multiple ramifications, avatars and connections. These include representation theory, quantum groups (when considering Hopf algebras instead of associative or Lie algebras), noncommutative geometry and manifolds, algebraic geometry (even algebraic curves à la Zagier), number theory, and of course what is regrouped under the name of M-theory. We shall here look at these from the unifying point of view of deformation theory and refer to a limited number of papers as a starting point for further study.

**This review is dedicated to Jerzy Plebański upon the occasion of his* 75^{th} *birthday.* Jerzy was, and still is, active in many fields, including those tackled in this review. I am especially grateful to him for being the first to mention me, in 1975 in Austin, TX, the then little known paper by Moyal. At the same time it is impossible not to associate this panorama with the ever present memory of Moshé Flato, the founder of, and a main player in, the field of deformation theory in view of physical applications, with whom I had the privilege to work as a team for thirty five years. It was thanks to Moshé and our common friend Ryszard Rączka that Jerzy and I became friends more than 30 years ago.

I. Introduction

A scientist should try and answer three questions: why, what and how. The bulk of the work is of course devoted to the last question: research is at best 10% inspiration and 90% perspiration. But it is better to know what one is doing, and it helps a lot to know why. The latter knowledge can often be imprecise and implicit, but very few works turn out to be important if the answer to the question "why?" is "why not?" – or if the research is merely solving a problem posed by some adviser or a guru, without asking oneself why is the problem important to solve.

Deformation theory provides a partial answer to the question "how". The knowledge of the category where one defines it clarifies what is done. We shall start this exposition by explaining why it is a powerful tool.

Physical theories have their domain of applicability defined by the relevant distances, velocities, energies, etc. involved. But the passage from one domain (of distances, etc.) to another does not happen in an uncontrolled way. Rather, experimental phenomena appear that cause a paradox and contradict accepted theories. Eventually a new fundamental constant enters and the formalism is modified. Then the attached structures (symmetries, observables, states, etc.) *deform* the initial structure. Namely, we have a new structure which in the limit, when the new parameter goes to zero, coincides with the previous formalism. The only question is, in which category do we seek for deformations? Usually physics is rather conservative and if we start e.g. with the category of associative or Lie algebras, we tend to deform in the same category, but there are important examples of generalizations of this principle (e.g. quantum groups are deformations of Hopf algebras).

The discovery of the non-flat nature of Earth may be the first example of this phenomenon. Closer to us, the paradox coming from the Michelson and Morley experiment (1887) was resolved in 1905 by Einstein with the special theory of relativity: in our context, one can express that by saying that the Galilean geometrical symmetry group of Newtonian mechanics is deformed to the Poincaré group, the new fundamental constant being c^{-1} where c is the velocity of light in vacuum.

It is interesting to note that a first mathematical example of deformations was introduced at around the same time with the Riemann surface theory, though deformations became systematically studied in the mathematical literature only at the end of the fifties with the profound works of Kodaira and Spencer [KS58] on deformations of complex analytic structures. Now, when one has an action on a geometrical structure, it is natural to try and "linearize" it by inducing from it an action on an algebra of functions on that structure. This is implicitly what Gerstenhaber did in 1963 [Ge63] with his definition and thorough study of deformations of rings and algebras.

It is in the Gerstenhaber sense that the Galileo group is deformed to the Poincaré group; that operation is the inverse of the notion of group contraction introduced ten years before, empirically, by Ínönü and Wigner [WI53]. This fact triggered strong interest for deformation theory in France among a number of theoretical physicists, including Flato who had just arrived from the Racah school and knew well the effectiveness of symmetry in physical problems. He was soon to realize that, however important symmetry is as a notion and a tool in a mathematical treatment of physical problems, it is not the only one and should be complemented with other (often related) concepts. The notion of deformation can be applied to a variety of categories that are used to express mathematically the physical reality.

One should not forget that mathematics arose as an abstraction of the physical world. Until the 19^{th} century, most leading mathematicians were also physicists, and vice-versa (sometimes also philosophers): Archemedes, Newton, Pascal, Laplace, Gauss, to mention just a few in the "Western World". In this respect Weyl was perhaps "the finest mind of the 19^{th} century", an expression diverted from the movie "Me and the Colonel" (starring Danny Kaye). In the middle of the 20^{th} century the two worlds became so widely separated by a kind of "Babel tower effect" that Wigner could marvel about the "unreasonable effectiveness of mathematics in theoretical physics". As Sir Michael Atiyah put it in his closing talk at ICMP 2000 (paraphrasing Oscar Wilde about the US and UK), "mathematics and physics became two communities separated by a common language." Many mathematicians were proud of knowing nothing about physics, and many physicists despise papers that are too mathematical for them (i.e. that they do not understand because they use mathematical notions which they ignore, or that painfully make as rigorous as possible handwaving physical arguments); every senior scientist can put famous names behind those attitudes, so I shall refrain from doing so.

In recent years the trend has been reversed, at least in mathematics. Somehow, among the variety of mathematical problems that the human brain can imagine, those having a physical origin tend to be most seminal – and mathematicians again realize that. The number of "quantum Fields medals", from Connes in 1982 to Kontsevich in 1998, is a clear proof of it. In physics the two trends coexist, not always peacefully, sometimes with exaggeration. Many physicists tend to agree with Goethe who said that "mathematicians are like Frenchmen, they translate everything into their own language and henceforth it is something entirely different." But more and more now realize the importance of being bilingual; as with languages, it is of consequence to learn both languages at a young age, because knowing facts is one thing and feeling them is another question. There are at present a few living examples of truly bilingual scientists, in all generations, but not enough.

In addition to, but not unrelated with, the importance of deformation theory in model building, to which the bulk of this paper is devoted, another fact played a major role in Flato's philosophy: space-time cannot be disconnected from truly fundamental models. Nowadays this may seem obvious, albeit with space-times that may have more than the traditional four dimensions, but insisting in keeping that direction of research open (see e.g. [FS65]) was considered heretical in the mid 60's when the mainstream would not tolerate anything but particle spectroscopy based on (phenomenological) unitary groups commuting with the Poincaré group and ignored the limitations of "theorems" proving that. In this spirit we were at that time led to look at the conformal group and to another group [FHS] involving two kinds of translations, vectorial and spinorial, applications of which to neutrino physics gave a prototype of the Wess-Zumino [WZ] Poincaré supersymmetry.

The need for modeling is as old as Science: more and more data are being collected and it is natural to try and put some order there. So from experimental data E one imagines a model M that can explain them. Eventually one can sometimes show that the model M can be derived from more fundamental principles, from a theory T. If new data $E_1 \supsetneq E$, are found that can also be derived from T, some (not more than 3) may earn a trip to Stockholm in December and the fame that goes with it. By a confusion (frequent in everyday life) between necessary and sufficient conditions, it is sometimes said that abstract entities involved in T were "directly observed" with the new data: what has been observed is in fact only a consequence of T in some model. [See e.g. in `http://www.nobel.se/physics/laureates/1999/press.html` the remark that

"This [the top] quark was observed directly for the first time in 1995 at the Fermilab in the USA," somewhat strange for a "particle" supposed to be confined and thus not directly observable.] The confusion is enhanced by the fact that our interpretation of the raw experimental data is made within existing models or theories, so that what we call an experimental result is, in fact, theory-dependent. But it often happens that with a larger data set E', the new data will not be easily cast in the existing model. Then there will be a need to develop a new model M', if possible following from a new theory T', that can explain everything observed so far. What happened in the past shows that deformation theory, developed in an appropriate context, can lead us to such "deformed" models and theories.

On the epistemological side, that approach is in line with Kant's philosophy – and with Spinoza's pantheistic views, according to which (in mathematical terms) the Universe is the representation of God, possibly unfaithful. As pointed out last summer in a weekly magazine by the French Minister of Education, the philosopher Luc Ferry, Kant was the first to invert the traditional concept that God created man from his own image. Kant, a deeply religious person, did not go as far as the (blasphematory) opposite, later reached e.g. by marxists, but his view was that one has to start with man's imperfection and try and reach from it God's perfection. Deformation theory provides us with a tool to do just that, starting with an imperfect description and deforming it into a less imperfect one.

Doing so one may discover that often "complicated is simpler": The richer (deformed) structures carry more information, and may exhibit a variety of properties that make them easier to tackle mathematically than their more degenerate special cases.

I shall not develop further these questions of history and philosophy of sciences and stick to the main subject of this presentation – but not without presenting some argument which shows that deformation goes back to the very beginning. In the beginning, so goes the Bible in a perfect example of the Italian expression "traduttore traditore" (the translator is a traitor), God created the heavens and the earth. In the original Hebrew version, he created "et hashamayim ve et haaretz". But the sky (shamayim, literally "there water") was created only the second day, when God also separated the land from the seas ("water here"). If I may try a daring interpretation, in the style of my supposed ancestor Rabbi Shlomo Itzhaki (better known by the acronym Rashi, the most famous commentator of the Bible, who lived in Champagne in the 11$^{\text{th}}$ century), in the beginning God created "there and here", i.e. he created the Deformation, that makes us pass from the primordial identity symmetry to the Galileo and then the Poincaré symmetry of special relativity ('here') to the realm of general relativity at cosmological distances ('there'). This interpretation may seem blasphemous to some, and in any case I do not take it very seriously, but I found it cute enough to share it in this panorama.

In this presentation I shall concentrate on a "trilogy" of interrelated subjects where the deformation insight permits significant progress. In his closing lecture of TH2002 in Paris, C.N. Yang described what he called the "three melodies" that dominated the development of physics in the twentieth century: quantization, symmetries and "the phase factor" – the latter being of course, dealing with gauge theories, his favorite. Interestingly those three tunes were already present – the latter in embryonic form – in Hermann Weyl's seminal book [We31] from 1928, i.e. quantization, group theory and covariant field equations (in particular Maxwell-Dirac, with their Abelian gauge). Missing there is the concept of deformation which, as we have shown already in the 70's, explains the essence of quantization. It also explains the passage from one kind of symmetry to another, e.g. from the symmetry of classical (Galilean) mechanics to that of special relativity and from the latter to Anti de Sitter when locally a tiny negative curvature is permitted. Finally one cannot do physics without looking at field equations, and the latter are usually covariant under a symmetry group; but fields interact, in particular (but not only) when operating a measurement, which brings in nonlinear field equations, a kind of deformation of the free equations when a coupling constant appears. Then one needs, there also, to take into account quantum effects, which brings us back to the problem of quantization. That is the basis of the trilogy pushed forward by Flato and coworkers since the 70's. I shall briefly survey it here, devoting more space to the quantization part of the trilogy.

II. Quantization is a deformation

II.1 From quantization to the invention of deformation quantization

The need for quantization appeared for the first time in 1900 when, faced with the impossibility to explain otherwise the black body radiation, Planck proposed the quantum hypothesis: the energy of light is not emitted continuously but in quanta proportional to its frequency. He wrote h for the proportionality constant which bears his name. This paradoxical situation got a beginning of a theoretical basis when, in 1905, Einstein came with the theory of the photoelectric effect – for which he was awarded the Nobel prize in 1922 (for 1921). Around 1920, Prince Louis de Broglie was introduced to the photoelectric effect, together with the Planck–Einstein relations and the theory of relativity, in the laboratory of his much older brother, Maurice duc de Broglie. This led him, in 1923, to his discovery of the duality of waves and particles, which he described in his celebrated Thesis published in 1925, and to what he called 'mécanique ondulatoire'. German and Austrian physicists, in particular, Hermann Weyl, Werner Heisenberg and Erwin Schrödinger, followed by Niels Bohr, transformed it into the quantum mechanics that we know, where the observables are operators in Hilbert spaces of wave functions – and were lead to its probabilistic interpretation that neither Einstein nor de Broglie were at ease with.

Intuitively, classical mechanics is the limit of quantum mechanics when $\hbar = \frac{h}{2\pi}$ goes to zero. But how can this be realized when in classical mechanics the observables are functions over phase space (a Poisson manifold) and not operators? The deformation philosophy promoted by Flato shows the way: one has to look for deformations of algebras of classical observables, functions over Poisson manifolds, and realize there quantum mechanics in an *autonomous* manner.

What we call "deformation quantization" relates to (and generalizes) what in the conventional (operatorial) formulation are the Heisenberg picture and Weyl's quantization procedure. In the latter [We31], starting with a classical observable $u(p,q)$, some function on phase space $\mathbb{R}^{2\ell}$ (with $p,q \in \mathbb{R}^{\ell}$), one associates an operator (the corresponding quantum observable) $\Omega(u)$ in the Hilbert space $L^2(\mathbb{R}^{\ell})$ by the following general recipe:

$$u \mapsto \Omega_w(u) = \int_{\mathbb{R}^{2\ell}} \tilde{u}(\xi,\eta)\exp(i(P.\xi + Q.\eta)/\hbar)w(\xi,\eta)\, d^{\ell}\xi d^{\ell}\eta \qquad (1)$$

where $\tilde{u}$ is the inverse Fourier transform of u, P_α and Q_α are operators satisfying the canonical commutation relations $[P_\alpha, Q_\beta] = i\hbar\delta_{\alpha\beta}$ ($\alpha,\beta = 1,...,\ell$), w is a weight function and the integral is taken in the weak operator topology. What is now called normal ordering corresponds to choosing the weight $w(\xi,\eta)=\exp(-\frac{1}{4}(\xi^2\pm\eta^2))$, standard ordering (the case of the usual pseudodifferential operators in mathematics) to $w(\xi,\eta)=\exp(-\frac{i}{2}\xi\eta)$ and the original Weyl (symmetric) ordering to $w=1$. An inverse formula was found shortly afterwards by Eugene Wigner [Wi32] and maps an operator into what mathematicians call its symbol by a kind of trace formula. For example Ω_1 defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2\ell})$ and Hilbert-Schmidt operators on $L^2(\mathbb{R}^{\ell})$ with inverse given by

$$u=(2\pi\hbar)^{-\ell}\,\mathrm{Tr}[\Omega_1(u)\exp((\xi.P+\eta.Q)/i\hbar)] \tag{2}$$

and if $\Omega_1(u)$ is of trace class one has $\mathrm{Tr}(\Omega_1(u))=(2\pi\hbar)^{-\ell}\int u\,\omega^\ell\equiv\mathrm{Tr}_\mathrm{M}(u)$, the "Moyal trace", where ω^ℓ is the (symplectic) volume dx on $\mathbb{R}^{2\ell}$. Numerous developments followed in the direction of phase-space methods, many of which are described in [AW70]. Of particular interest to us here is the question of finding an interpretation to the classical function u, symbol of the quantum operator $\Omega_1(u)$; this was the problem posed (around 15 years after [Wi32]) by Blackett to his student Moyal. The (somewhat naive) idea to interpret it as a probability density had of course to be rejected (because u has no reason to be positive) but, looking for a direct expression for the symbol of a quantum commutator, Moyal found [Mo49] what is now called the Moyal bracket:

$$M(u_1,u_2)=\nu^{-1}\sinh(\nu P)(u_1,u_2)=P(u_1,u_2)+\sum_{r=1}^{\infty}\frac{\nu^{2r}}{(2r+1)!}P^{2r+1}(u_1,u_2) \tag{3}$$

where $2\nu=i\hbar$, $P^r(u_1,u_2)=\Lambda^{i_1j_1}\ldots\Lambda^{i_rj_r}(\partial_{i_1\ldots i_r}u_1)(\partial_{j_1\ldots j_r}u_2)$ is the r^{th} power ($r\geq 1$) of the Poisson bracket bidifferential operator P, $i_k,j_k=1,\ldots,2\ell$, $k=1,\ldots,r$ and $(\Lambda^{i_kj_k})=\binom{0\,-I}{I\;\;0}$. To fix ideas we may assume here $u_1,u_2\in C^\infty(\mathbb{R}^{2\ell})$ and the sum is taken as a formal series (the definition and convergence for various families of functions u_1 and u_2 was also studied, including in [BFFLS]). A similar formula for the symbol of a product $\Omega_1(u)\Omega_1(v)$ had been found a little earlier [Gr46] and can now be written more clearly as a (Moyal) *star product*:

$$u_1 *_M u_2=\exp(\nu P)(u_1,u_2)=u_1u_2+\sum_{r=1}^{\infty}\frac{\nu^r}{r!}P^r(u_1,u_2). \tag{4}$$

Several integral formulas for the star product have been introduced and the Wigner image of various families of operators (including bounded operators on $L^2(\mathbb{R}^\ell)$) were studied. It was noticed, however after deformation quantization was introduced, that the composition of symbols of pseudodifferential operators (ordered, like differential operators, "first q, then p") is a star product. Starting from field theory, where normal (Wick) ordering is essential (the role of q and p above is played by $q\pm ip$), Berezin [Be74] developed in the mid-seventies an extensive study of what he called "quantization", based on the correspondence principle and Wick symbols. It is essentially based on Kähler manifolds and related to pseudodifferential operators in the complex domain [BG81]. However in his theory (which we noticed rather late), as in the studies of various orderings [AW70], the important concepts of *deformation* and *autonomous* formulation of quantum mechanics in general phase space are absent.

Quantization involving more general phase spaces was treated, in a somewhat systematic manner, only with Dirac constraints [Di]: second class Dirac constraints restrict phase space from some

$\mathbb{R}^{2\ell}$ to a symplectic manifold W imbedded in it (with induced symplectic form), while first class constraints further restrict to a Poisson manifold with symplectic foliation (see e.g. [FLS76]). The question of quantization on such manifolds was certainly treated by many authors (including [Di]) but did not go beyond giving some (often useful) recipes and hoping for the best.

A first systematic attempt started around 1970 with what was called soon afterwards *geometric quantization* [Kt70], a by-product of Lie group representations theory where it gave significant results. It turns out that it is geometric all right, but its scope as far as quantization is concerned has been rather limited since few classical observables could be quantized, except in situations which amount essentially to the Weyl case considered above. In a nutshell one considers phase-spaces W which are coadjoint orbits of some Lie groups (the Weyl case corresponds to the Heisenberg group with the canonical commutation relations and $\mathfrak{h}_\ell$ as Lie algebra); there one defines a "prequantization" on the Hilbert space $L^2(W)$ and tries to halve the number of degrees of freedom by using polarizations (often complex ones, which is not an innocent operation as far as physics is concerned) to get a Lagrangean submanifold $\mathscr{L}$ of dimension half of that of W and quantized observables as operators in $L^2(\mathscr{L})$. A recent exposition can be found in [Wo92]. Since physicists have no problem with quantizing classical observables (at least in flat space), there was clearly a practical gap that needed to be filled. From the conceptual point of view, the "quantum jump" in the nature of observables required also an explanation. The answer to both questions was given by deformation quantization, which was reviewed recently more in details in [DS02, St98].

Before we get into the heart of the deformation aspect of quantization, a mathematical parenthesis on deformation theory, in general and in the differentiable context, is needed. For convenience of the exposition I shall also include there a very brief account of the fundamental mathematical problem of the existence and classification of star products on general symplectic (solved in 1983 – 1994) and on Poisson manifolds (solved in 1997), plus some words on the recent extension to algebraic varieties. These metamorphoses of deformation quantization are related to some of the deepest recent mathematical developments. Quantum groups can be viewed as an avatar of deformation quantization, when the category in question is that of Hopf algebras; we shall briefly explain how. Noncommutative geometry is very much related to deformation quantization in several respects; some are presented in Connes' book [Co]; a very elaborate beginning of the theory of noncommutative manifolds (especially in dimension 4) can be found in [CDV01].

The reader only interested only in the physical aspects can just glance through that subsection and be satisfied with knowing it exists.

II.2 Cohomologies, deformation theory, examples and new developments

In an often ignored section of a paper, I.E. Segal [Se51] and (independently) a little later Wigner and Ínönü [WI53, Sa61] have introduced in the early fifties a kind of inverse [LN67] to the mathematical notion of *deformation* of Lie groups and algebras, notion which was precisely defined only in 1964 by Murray Gerstenhaber [Ge63]. That inverse was called *contraction*. Typical examples (mentioned at the beginning of this Section) are the passage from de Sitter to Poincaré groups (taking the limit of zero curvature in space-time) or from Poincaré to Galilei (taking $c^{-1} \to 0$). Intuitively speaking a contraction is performed by neglecting in symmetries, at some level of physical reality, a constant (like c^{-1}) which has negligible impact at this level but significant effects at a more "refined" level.

This may be realized mathematically in varying generality (e.g. [Sa61] is more general than [WI53] but both have for inverse a Gerstenhaber deformation). The notion of deformation of algebras may be seen as a sort of linearized version of the notion (introduced a few years before, mainly by Kodaira and Spencer [KS58]) of deformations of complex analytic structures. Its mathematical theory is better defined than that of contractions. For completeness I shall now present it briefly. All this is by now well known, even to many physicists. Therefore I keep details to a minimum, referring the interested reader to papers and textbooks cited here and references quoted therein.

We have recently (see [DFST, Pi97, Na98]) introduced deformations which are more general than those introduced by Gerstenhaber, in order e.g. to quantize Nambu mechanics. They are still inverse to some contraction procedure, applied to algebras which are not geometrical symmetries. I shall not enter here into more details but note that one should keep an open mathematical mind, let physics be a guide and develop when needed new mathematical tools.

II.2.1 Hochschild and Chevalley-Eilenberg cohomologies. Let first A be an *associative* algebra (over some commutative ring $\mathbb{K}$) and for simplicity I consider it as a module over itself with the adjoint action (algebra multiplication); the generalization to cohomology valued in a general module is straightforward. A *p-cochain* is a p-linear map C from A^p into (the module) A and its *coboundary* *bC* is given by

$$\begin{aligned} bC(u_0,\dots,u_p) &= u_0C(u_1,\dots,u_p) - C(u_0u_1,u_2,\dots,u_p) + \cdots \\ &+ (-1)^pC(u_0,u_1,\dots,u_{p-1}u_p) + (-1)^{p+1}C(u_0,\dots,u_{p-1})u_p. \end{aligned} \tag{5}$$

One checks that we have here what is called a complex, i.e. $b^2 = 0$. We say that a p-cochain C is a *p-cocycle* if $bC = 0$. We denote by $\mathscr{Z}^p(A,A)$ the space of p-cocycles and by $\mathscr{B}^p(A,A)$ the space of those p-cocycles which are coboundaries (of a $(p-1)$-cochain). The pth *Hochschild cohomology* space (of A valued in A) is defined [Ho45] as $\mathscr{H}^p(A,A) = \mathscr{Z}^p(A,A)/\mathscr{B}^p(A,A)$. *Cyclic cohomology* [Co] is defined using a bicomplex which includes the Hochschild complex and is important e.g. when traces have to be used (see closed star products below).

For *Lie algebras* (with bracket $\{\cdot,\cdot\}$) one has a similar definition, due to Chevalley and Eilenberg [CE48]. The p-cochains are here skew-symmetric, i.e. linear maps $B : \wedge^p A \longrightarrow A$, and the Chevalley coboundary operator ∂ is defined on a p-cochain B by (where $\hat{u}_j$ means that u_j has to be omitted):

$$\begin{aligned} \partial C(u_0,\dots,u_p) = \ & \textstyle\sum_{j=0}^p (-1)^j\{u_j, C(u_0,\dots,\hat{u}_j,\dots,u_p)\} \\ &+ \sum_{i<j}(-1)^{i+j}C(\{u_i,u_j\},u_0,\dots,\hat{u}_i,\dots,\hat{u}_j,\dots,u_p). \end{aligned} \tag{6}$$

Again one has a complex ($\partial^2 = 0$), cocycles and coboundaries spaces Z^p and B^p (resp.) and by quotient the *Chevalley cohomology* spaces $H^p(A,A)$, or in short $H^p(A)$; the collection of all cohomology spaces is often denoted H^*.

II.2.2 Gerstenhaber theory of deformations of algebras. Let A be an algebra. By this we mean an *associative, Lie* or *Hopf* algebra, or a *bialgebra*. Whenever needed we assume it is also a *topological* algebra, i.e. endowed with a locally convex topology for which all needed algebraic laws are continuous. For simplicity we may think that the base (commutative) ring $\mathbb{K}$ is the field of complex numbers $\mathbb{C}$ or that of the real numbers $\mathbb{R}$. Extending it to the ring $\mathbb{K}[[\nu]]$ of formal series in some parameter ν gives the module $\tilde{A} = A[[\nu]]$, on which we can consider the preceding various algebraic (and topological) structures.

Deformations and cohomologies. A concise formulation of a Gerstenhaber deformation of an algebra (which we shall call in short a *DrG-deformation* whenever a confusion may arise with more general deformations) is [Ge63, GS88, BFGP]:

DEFINITION 1 *A deformation of such an algebra A is a $\mathbb{K}[[\nu]]$-algebra $\tilde{A}$ such that $\tilde{A}/\nu\tilde{A} \approx A$. Two deformations $\tilde{A}$ and $\tilde{A}'$ are said equivalent if they are isomorphic over $\mathbb{K}[[\nu]]$ and $\tilde{A}$ is said trivial if it is isomorphic to the original algebra A considered by base field extension as a $\mathbb{K}[[\nu]]$-algebra.*

Whenever we consider a topology on A, $\tilde{A}$ is supposed to be topologically free. For associative (resp. Lie) algebras, Definition 1 tells us that there exists a new product $*$ (resp. bracket $[\cdot,\cdot]$) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. $\{\cdot,\cdot\}$) this means that, for $u_1, u_2 \in A$ (we can extend this to $A[[\nu]]$ by $\mathbb{K}[[\nu]]$-linearity) we have:

$$u_1 * u_2 = u_1 u_2 + \sum_{r=1}^{\infty} \nu^r C_r(u_1, u_2) \tag{7}$$

$$[u_1, u_2] = \{u_1, u_2\} + \sum_{r=1}^{\infty} \nu^r B_r(u_1, u_2) \tag{8}$$

where the C_r are Hochschild 2-cochains and the B_r (skew-symmetric) Chevalley 2-cochains, such that for $u_1, u_2, w \in A$ we have $(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$ and $\mathscr{S}[[u_1, u_2], u_3] = 0$, where $\mathscr{S}$ denotes summation over cyclic permutations. At each level r we therefore need to fulfill the equations $(j, k \geq 1)$:

$$D_r(u_1, u_2, u_3) \equiv \sum_{j+k=r} (C_j(C_k(u_1, u_2), u_3) - C_j(u_1, C_k(u_2, u_3)) = bC_r(u_1, u_2, u_3) \tag{9}$$

$$E_r(u_1, u_2, u_3) \equiv \sum_{j+k=r} \mathscr{S} B_j(B_k(u_1, u_2), u_3) = \partial B_r(u_1, u_2, u_3) \tag{10}$$

where b and ∂ denote (respectively) the Hochschild and Chevalley coboundary operator. In particular we see that for $r = 1$ the driver C_1 (resp. B_1) must be a 2-cocycle. Furthermore, assuming one has shown that (9) or (10) are satisfied up to some order $r = t$, a simple calculation shows that the left-hand sides for $r = t + 1$ are then 3-cocycles, depending only on the cochains C_k (resp. B_k) of order $k \leq t$. If we want to extend the deformation up to order $r = t + 1$ (i.e. to find the required 2-cochains C_{t+1} or B_{t+1}), this cocycle has to be a coboundary (the coboundary of the required cochain): *The obstructions to extend a deformation from one step to the next lie in the 3-cohomology*. In particular (and this was Vey's trick) if one can manage to always pass through the null class in the 3-cohomology, a cocycle can be the driver of a full-fledged (formal) deformation.

For a (topological) *bialgebra* (an associative algebra A where we have in addition a coproduct $\Delta : A \longrightarrow A \otimes A$ and the obvious compatibility relations), denoting by $\otimes_\nu$ the tensor product of $\mathbb{K}[[\nu]]$-modules, we can identify $\tilde{A} \hat{\otimes}_\nu \tilde{A}$ with $(A \hat{\otimes} A)[[\nu]]$, where $\hat{\otimes}$ denotes the algebraic tensor product completed with respect to some topology (e.g. projective for Fréchet nuclear topology), we similarly have a deformed coproduct $\tilde{\Delta} = \Delta + \sum_{r=1}^{\infty} \nu^r D_r$, $D_r \in \mathscr{L}(A, A\hat{\otimes}A)$ and in this context appropriate cohomologies can be introduced. Here we shall not elaborate on these, nor on the additional requirements for Hopf algebras, referring for more details to original papers and books. There is a huge literature on the subject, especially for algebras of functions on Poisson-Lie groups and their topological duals, among which we shall only mention [Ji85, Dr86, GS88, BFGP, SS93, ES98] and references quoted therein.

Equivalence means that there is an isomorphism $T_\nu = I + \sum_{r=1}^{\infty} \nu^r T_r$, $T_r \in \mathscr{L}(A,A)$ so that $T_\nu(u_1 *' u_2) = (T_\nu u_1 * T_\nu u_2)$ in the associative case, denoting by $*$ (resp. $*'$) the deformed laws in $\tilde{A}$ (resp. $\tilde{A}'$); and similarly in the Lie case. In particular we see (for $r = 1$) that a deformation is trivial at order 1 if it starts with a 2-cocycle which is a 2-coboundary. More generally, exactly as above, we can show [BFFLS] that if two deformations are equivalent up to some order t, the condition to extend the equivalence one step further is that a 2-cocycle (defined using the T_k, $k \le t$) is the coboundary of the required T_{t+1} and therefore *the obstructions to equivalence lie in the 2-cohomology*. In particular, if that space is null, all deformations are trivial.

Unit. An important property is that a *deformation of an associative algebra with unit* (what is called a unital algebra) is again unital, and *equivalent to a deformation with the same unit*. This follows from a more general result of Gerstenhaber (for deformations leaving unchanged a subalgebra) and a proof can be found in [GS88].

Hopf algebras. In the case of (topological) *bialgebras* or *Hopf* algebras, *equivalence* of deformations has to be understood as an isomorphism of (topological) $\mathbb{K}[[\nu]]$-algebras, the isomorphism starting with the identity for the degree 0 in ν. A deformation is again said *trivial* if it is equivalent to that obtained by base field extension. For Hopf algebras the deformed algebras may be taken (by equivalence) to have the same unit and counit, but in general not the same antipode.

II.2.3 Examples of special interest: the differentiable cases. Consider the algebra $N = C^\infty(X)$ of functions on a differentiable manifold X. When we look at it as an associative algebra acting on itself by pointwise multiplication, we can define the corresponding Hochschild cohomologies. Now let Λ be a skew-symmetric contravariant two-tensor (possibly degenerate) defined on X, satisfying $[\Lambda, \Lambda] = 0$ in the sense of the supersymmetric Schouten-Nijenhuis bracket (a definition of which, both intrinsic and in terms of local coordinates, can be found in [BFFLS, FLS74]). Then the inner product $P(u_1, u_2) = i(\Lambda)(du_1 \wedge du_2)$ of Λ with the 2-form $du_1 \wedge du_2$, $u_1, u_2 \in N$, defines a *Poisson bracket P*: it is obviously skew-symmetric, satisfies the Jacobi identity because $[\Lambda, \Lambda] = 0$ and the Leibniz rule $P(u_1 u_2, u_3) = P(u_1, u_3)u_2 + u_1 P(u_2, u_3)$. It is a bidifferential 2-cocycle for the (general or differentiable) Hochschild cohomology of N, skewsymmetric of order $(1,1)$, therefore [BFFLS] nontrivial and thus defines an infinitesimal deformation of the pointwise product on N. We say that X, equipped with such a P, is a *Poisson manifold* [BFFLS, Li77].

When Λ is everywhere nondegenerate (X is then necessarily of even dimension 2ℓ), its inverse ω is a closed everywhere nondegenerate 2-form ($d\omega = 0$ is then equivalent to $[\Lambda, \Lambda] = 0$) and we say that X is *symplectic*; ω^ℓ is a volume element on X. Then one can in a consistent manner work with differentiable cocycles [BFFLS, FLS74] and the differentiable Hochschild p-cohomology space $\mathscr{H}^p(N)$ is [HKR, Ve75] that of all skew-symmetric contravariant p-tensor fields, and therefore is infinite-dimensional. Thus, except when X is of dimension 2 (because then necessarily $\mathscr{H}^3(N) = 0$), the obstructions belong to an infinite-dimensional space where they may be difficult to trace. On the other hand, when $2\ell = 2$, any 2-cocycle can be the driver of a deformation of the associative algebra N: "anything goes" in this case; some examples for $\mathbb{R}^2$ can be found in [Ve75].

Now endow N with a Poisson bracket: we get a Lie algebra and can look at its Chevalley cohomology spaces. Note that P is bidifferential of order $(1,1)$ so it is important to check whether the Gerstenhaber theory is *consistent* when restricted to *differentiable* cochains (both of arbitrary order and of order at most 1). This is a nontrivial question and we gave it a positive answer [FLS74, BFFLS]; in brief, if a coboundary is differentiable, it is the coboundary of a differentiable cochain.

Again, since P is of order (1,1), we first studied the 1-differentiable cohomologies. When the cochains are restricted to be of order $(1,1)$ with no constant term (then they annihilate constant functions, which we write "n.c." for "null on constants") it was found [Li74] that the Chevalley-Eilenberg cohomology $H^*_{1-\text{diff,n.c.}}(N)$ of the Lie algebra N (acting on itself with the adjoint representation) is exactly the de Rham cohomology $H^*_{dR}(X)$. Thus $\dim H^p_{1-\text{diff,n.c.}}(N) = b_p(X)$, the pth Betti number of the manifold X. Without the n.c. condition one gets a slightly more complicated formula [Li74]; in particular if X is symplectic with an exact 2-form $\omega = d\alpha$, one has here $H^p_{1-\text{diff}}(N) = H^p_{dR}(X) \oplus H^{p-1}_{dR}(X)$.

This allowed us to study in 1974 what we called in [FLS74] *1-differentiable deformations* of the Poisson bracket Lie algebra N and to give some applications [FLS76]. In particular we noticed that the "pure" order $(1,1)$ deformations correspond to a deformation of the 2-tensor Λ. Shortly afterwards, triggered by our works, J. Vey [Ve75] noticed that in fact $\dim H^2_{\text{diff,n.c.}}(N) = 1 + \dim H^2_{1-\text{diff,n.c.}}(N)$ and that $H^3_{\text{diff,n.c.}}(N)$ is also finite-dimensional, which enabled him to study differentiable deformations. Incidentally, in the $\mathbb{R}^{2\ell}$ case, Vey rediscovered (independently, because he ignored it) the Moyal bracket. The latter was then rather "exotic" and few authors (except for a number of physicists, like [AW70], J. Plebański who described it in Polish lecture notes [Pl69] he gave us later, and L. Faddeev who had mentioned it in his lectures in Leningrad around the same time) paid any attention to it. In *Mathematical Reviews* this bracket, for which [Mo49] is nowadays often quoted, is not even mentioned in the review! We then came back to the problem [FLScr], this time with differentiable deformations, and deformation quantization was conceived.

II.2.4 Existence, classification, metamorphoses and avatars.

The symplectic case. As early as 1975, Vey [Ve75] had shown that on a symplectic manifold X with $b_3(X) = 0$, there exists a globally defined deformation of the Poisson bracket P. He did this by a careful study allowing him to show (by induction) that at each order of deformation (each order of $\hbar$), he can manage to pass via the zero class of the finite-dimensional obstructions space $H^3_{\text{diff,n.c.}}(N)$. This was later easily extended to star products and in essence tells us that we can in this case "glue" Moyal products defined on local charts (equivalence will take care of intersection of two charts and the vanishing of $b_3(X)$ permits to do it in a way compatible with multiple intersections).

The restriction $b_3(X) = 0$ was purely technical and indeed already in [BFFLS], with the important case of the hydrogen atom ($X = T^*(S^3)$), we showed that it is not essential. The latter case was generalized by Gutt to $X = T^*(M)$ where M is a Lie group or more generally a parallelizable manifold. De Wilde and Lecomte were shortly afterwards able to find quite abstract proofs of existence in increasing generality, in 1983 [DL83] for a general symplectic manifold. What is behind the scene for cotangent bundles $T^*(M)$ is that one has globally defined "momentum coordinates" which permits to work with globally defined differentiations on M and polynomials in them (differential operators). From there a natural step forward is to localize everything on a general symplectic manifold X and to work with a bundle $\mathscr{W}(X)$ of *Weyl algebras*; a Weyl algebra W_ℓ is the enveloping algebra of the Heisenberg canonical commutation relations Lie algebra $\mathfrak{h}_\ell$, possibly completed to formal series, and the product there is the $\mathrm{Sp}(2\ell)$-invariant Moyal product. The "miracle" is that this bundle has a flat connection and a global section; therefore locally defined (Moyal) star products on polynomials (in any given Darboux chart) can be glued together to a globally defined star product. (A Darboux chart on a symplectic manifold is a chart with local coordinates the q's and p's of physicists, i.e. where $\omega = \sum_{j=1}^{2\ell} dq_j \wedge dp_j$.)

This line of conduct was taken by Fedosov since 1985 (in an obscure paper which was made detailed and precise later [Fe94, Fe96]), using a "germ" approach (infinitesimal neighborhoods), symplectic connections and an algorithmic construction of the flat connection on the Weyl bundle, canonically constructed starting from a given symplectic connection on X. Independently a Japanese group [OMY] obtained also a general proof of existence, gluing together Moyal products defined on Darboux charts by "projecting" from the Weyl bundle; the method could also be easily adapted to give the existence of *closed star products* [CFS92], important because they permit to consider generalized traces, and generalized to some Poisson manifolds. For more details on these (especially Fedosov's) constructions we refer to the original papers and to the reviews in [BCG, GR99, Ws94]. The relation between the "Russian" approach and the "Belgian" one was made [De95] by a famous Belgian mathematician (with a Russian wife), translating both into his own language of "gerbes". Eventually the Fedosov construction was shown (cf. e.g. [NT95]) to be "generic" in the sense that any differentiable star product is equivalent to a Fedosov star product (and to a Vey product [Xu98]).

An important tool in symplectic mechanics is that of *reduction* [MW74] caused by the action of an invariance group G and subsequent reduction of the algebra of observables. In fact, already in [BFFLS], a reduction of this general type was used in connection with the hydrogen atom. Fedosov has recently showed [Fe98] that the classical reduction theory can be "quantized" in the same conditions, i.e. that reduction commutes with G-invariant deformation quantization (at least for G compact); note however that, as shown in a simple example [Wa98], reductions may give nonequivalent star products.

The equivalence classes of star products are in one-to-one correspondence with formal series in the deformation parameter ν with coefficients in $H^2_{dR}(X)$, i.e. series $[\Lambda] + \nu[H^2_{dR}(X)][[\nu]]$, where Λ is the (nondegenerate) Poisson 2-tensor given on X and $[\Lambda]$ is the de Rham class of the corresponding 2-form. Since $H^2_{dR}(X)$ classifies 1-differentiable deformations of the Lie algebra (N,P), one expects that *any Fedosov deformation can be obtained by a sequence of successive 1-differentiable deformations of the initial Poisson bracket*, starting with any given one, at least for X symplectic. This is indeed true [Bo98]. So by "plugging in" 1-differentiable deformations of Poisson brackets one can cover all possible equivalence classes, starting from any given Fedosov star product. In a more abstract form a similar conclusion can be rather easily derived as a consequence of the results developed in the nice review [GR99]. The mathematically oriented reader will find there a detailed presentation (in a Čech cohomology context) of the equivalence question and related problems.

Poisson finite-dimensional manifolds. As we explained earlier, physics (e.g. with first class Dirac constraints) requires sometimes manifolds which have a Poisson bracket P, but a degenerate one and are therefore not symplectic. These are called *Poisson manifolds* [BFFLS] and they are foliated with symplectic leaves. A typical example is the dual of a finite-dimensional Lie algebra, foliated by coadjoint orbits (see e.g. [Ki76]). As this example shows (even in the case of the Heisenberg Lie algebra $\mathfrak{h}_\ell$, where one must not forget the trivial orbit), Poisson manifolds are in general not regular; a *regular* Poisson manifold is foliated with symplectic leaves of constant even dimension (before introducing Poisson manifolds we had considered [FLS74] "canonical manifolds", regular Poisson manifolds where the leaves have codimension 1). In the regular case the theory we just explained extends in a straightforward manner.

The solution to this difficult problem was given by Maxim Kontsevich [Ko97]. It involves very elaborate constructions, both conceptually and computationally and makes an essential use of ideas coming from string theory. We shall not attempt here to describe it in detail but give the flavor of the development.

Since, as we noted above, a Poisson bracket P is a nontrivial 2-cocycle for the Hochschild cohomology of the algebra $N = C^\infty(X)$, a natural question is to decide whether this infinitesimal deformation can be extended to a star product on N. Kontsevich answers positively, and more:

THEOREM 1 *Let X be a differentiable manifold and $N = C^\infty(X)$. There is a natural isomorphism between equivalence classes of deformations of the null Poisson structure on X and equivalence classes of differentiable deformations of the associative algebra N; in particular, any Poisson bracket P on X comes from a canonically defined (modulo equivalence) star product.*

With this concise formulation of the result (the proof of which relies on giving a positive answer to Kontsevich's formality conjecture [Ko96]) we see that, in this more general context, a main result from the symplectic case is still valid: *classes of star products correspond to classes of deformations of the Poisson structure*. A deformation of the null Poisson structure is a formal series $\Lambda(\hbar) = \sum_{n=1}^{\infty} \Lambda_n \hbar^n$ having vanishing Schouten-Nijenhuis bracket with itself: $[\Lambda(\hbar), \Lambda(\hbar)] = 0$. Any given Poisson structure Λ_0 can be identified with the series $\Lambda_0\hbar$.

The bulk of the original proof of [Ko97] is the "affine case", essentially when X is some $\mathbb{R}^\ell$ (or an open set in it), the result being formulated in such a way that "gluing" charts, though still nontrivial, is not too difficult to perform for an experienced mathematician. Doing so Kontsevich gives an interesting *explicit* universal formula for the star product on such an X where graphs (and Stokes formula) play a crucial role. The formula can be written $u_1 * u_2 = \sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma,\Lambda}(u_1, u_2)$ where Λ is an arbitrary Poisson structure on an open domain in $\mathbb{R}^\ell$ and G_n a set of labeled oriented graphs Γ. The latter are pairs (V_Γ, E_Γ) such that $E_\Gamma \subset V_\Gamma \times V_\Gamma$ with $n+2$ vertices V_Γ and $2n$ edges E_Γ satisfying some additional conditions (see [Ko97], section 2). G_n has $(n(n+1))^n$ elements (1 for $n = 0$). B_Γ is an explicitly defined bidifferential operator (of total order $2n$, as in (4)) and w_Γ a weight defined by an absolutely convergent integral of the exterior product of the differentials of $2n$ harmonic angular variables associated with Γ, taken over the space of configurations of n numbered pairwise distinct points on the Lobatchevsky upper half-plane. These weights are examples of a very interesting class of algebraic numbers, called "periods" by Kontsevich [KoSMF]. In his Berlin

August 25, 1998 "Fields medal" lecture, *Motivic Galois group and deformation quantization*, Kontsevich stressed that the isomorphism described in Theorem 1 should be taken as one of a family of isomorphisms and indicates that the motivic Galois group should act on the moduli space of noncommutative algebras. This idea was developed extensively in his subsequent papers [Ko99, KoS], being made more precise thanks to the introduction (by Tamarkin in 1998 [Ta98]) of operads into the picture. Indeed, quoting from the introduction of [KoS]:

"The deformation theory of associative algebras is a guide for developing the deformation theory of many algebraic structures. Conversely, all the concepts of what should be the "deformation theory of everything" must be tested in the case of associative algebras. An associative algebra is an algebra over an operad. This fact, along with the observation that in many examples we are dealing with algebras over operads, "explains" the universality of associative algebras. It also suggests how to develop the deformation theory of algebras over operads... Another remarkable fact is the relationship of the deformation theory of associative algebras to the geometry of configuration spaces of points on surfaces. One of its incarnations is the Deligne conjecture. A brief history of the conjecture as well as its generalizations can be found in [Ko99]...The theme which motivated the whole project is the Grothendieck-Teichmüller group and its role in deformation theory."

Recently Kontsevich accomplished still another leap forward [Ko01] by going from the C^∞ situation to the algebro-geometric setting, performing what can be called 'noncommutative algebraic geometry'. In the previous case, operads and deformations of algebras over operads were essential notions. In the new setting he had to use sheaves of algebroids, a slightly more complicated structure. Both developments have a wide array of ramifications and implications in a variety of frontier mathematics. These metamorphoses of our original deformation quantization program are thus intimately related with frontier mathematical developments. We refer to the quoted literature for precise results.

II.3 Physics with deformation quantization

In our view, deformation quantization is not merely "a reformulation of the problem of quantizing a classical mechanical system" ([DN01], Sect. VI.G). It is much more, and is probably from the conceptual point of view the most nontrivial paradigm of Flato's deformation philosophy: *quantization is a deformation.* The physical reality is quantized even if, in most "classical" domains, the quantum effects are so small that they can be neglected in order to get a more practical mathematical description of reality. This was the case until the 19^{th} century; on the basis of the physical reality of that time, quantization and relativity appear as deformations. As we shall see below in some examples, the main results of conventional quantum theory can be derived in an *autonomous* manner within deformation quantization, without need of the operator formulation. The correspondence between both, when it is possible (via appropriate Weyl or Wigner maps), shows the consistency of our point of view with the conventional one. In these cases, it is a matter of practical choice to compute with one approach or the other. But deformation quantization works in cases when the traditional framework is not well (or not at all) adapted, which justifies our point of view. Interestingly, that openness to new points of view on quantum mechanics can already be found in [Di49].

II.3.1 Harmonic oscillator, angular momentum and hydrogen atom. Already in the founding papers [BFFLS] we have shown, by a number of practical examples (in effect, those for which a complete and rigorous spectral theory exists), that the advantage of our approach is not only conceptual.

Spectrality. Physicists want to get numbers matching experimental results, e.g. for energy levels of a system. That is usually achieved by describing the spectrum of a given Hamiltonian $\hat{H}$ supposed to be a self-adjoint operator so as to get a real spectrum and so that the evolution operator (the exponential of $it\hat{H}$) is unitary (thus preserves probability). A similar spectral theory can be done here, in an *autonomous manner*. The most efficient way to achieve it is to consider [BFFLS] the *star exponential* (corresponding to the evolution operator)

$$\mathrm{Exp}(Ht) \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{i\hbar}\right)^{n}(H*)^{n} \tag{11}$$

where $(H*)^n$ means the n^{th} star power of the Hamiltonian $H \in N$ (or $N[[\nu]]$). Then one writes its Fourier-Stieltjes transform $d\mu$ (in the distribution sense) as $\mathrm{Exp}(Ht) = \int e^{\lambda t/i\hbar} d\mu(\lambda)$ and defines *the spectrum of* $(H/\hbar)$ as the *support* S *of* $d\mu$ (incidentally this is the definition given by L. Schwartz for the spectrum of a distribution, out of motivations coming from Fourier analysis). In the particular case where H has discrete spectrum, the integral can be written as a sum (see the top equation in (13) below for a typical example): the distribution $d\mu$ is a sum of "delta functions" supported at the points of S multiplied by the symbols of the corresponding eigenprojectors.

In different orderings with various weight functions w in (1) one gets in general different operators for the same classical observable H, thus different spectra. For $X = \mathbb{R}^{2\ell}$ all orderings are mathematically equivalent (to Moyal under the Fourier transform T_w of the weight function w). This means that every observable H will have the same spectrum under Moyal ordering as T_wH under the equivalent ordering. But this does not imply physical equivalence, i.e. the fact that H will have the same spectrum under both orderings. On the contrary [CFGS]: if two equivalent star products are isospectral (give the same spectrum for a large family of observables and all $\hbar$), they are identical.

It is worth mentioning that our definition of spectrum permits to define a spectrum even for symbols of non-spectrable operators, such as the derivative on a half-line which has different deficiency indices; this corresponds to an infinite potential barrier. That is one of the many advantages of our autonomous approach to quantization.

Applications. In quantum mechanics it is preferable to work (for $X = \mathbb{R}^{2\ell}$) with the Moyal product, which has maximal symmetry, i.e. has $\mathfrak{sp}(\mathbb{R}^{2\ell}) \cdot \mathfrak{h}_\ell$ as what we call (see II.3.2 below) algebra of preferred observables. One indeed finds [BFFLS] that star powers of these preferred observables H (polynomials of order ≤ 2) are usual polynomials in H (not only in p and q), and as a consequence their star exponential is proportional to the usual exponential and a function of H. More precisely, if $H = \alpha p^2 + \beta pq + \gamma q^2 \in \mathfrak{sl}(2)$ with $p, q \in \mathbb{R}^\ell$, $\alpha, \beta, \gamma \in \mathbb{R}$, setting $d = \alpha\gamma - \beta^2$ and $\delta = |d|^{1/2}$ one gets by summing the star exponential and then taking its Fourier (or Fourier-Stieltjes) development, the sums and integrals appearing in the various expressions of the star exponential being convergent as distributions, in phase-space variables and in t or λ:

$$\operatorname{Exp}(Ht) = \begin{cases} (\cos\delta t)^{-l}\exp((H/i\hbar\delta)\tan(\delta t)) & \text{for} \quad d>0 \\ \exp(Ht/i\hbar) & \text{for} \quad d=0 \\ (\cosh\delta t)^{-l}\exp((H/i\hbar\delta)\tanh(\delta t)) & \text{for} \quad d<0 \end{cases} \tag{12}$$

$$\operatorname{Exp}(Ht) = \begin{cases} \sum_{n=0}^{\infty}\Pi_n^{(\ell)}e^{(n+\frac{\ell}{2})t} & \text{for} \quad d>0 \\ \int_{-\infty}^{\infty}e^{\lambda t/i\hbar}\Pi(\lambda,H)d\lambda & \text{for} \quad d<0 \end{cases} \tag{13}$$

We thus get the discrete spectrum $(n+\frac{\ell}{2})\hbar$ of the *harmonic oscillator* and the continuous spectrum $\mathbb{R}$ for the dilation generator pq. The eigenprojectors $\Pi_n^{(\ell)}$ and $\Pi(\lambda,H)$ are given [BFFLS] by known special functions on phase-space (generalized Laguerre and hypergeometric, multiplied by some exponential). Formulas (12) and (13) can, by analytic continuation, be given a sense outside singularities and even (as distributions) for values of t for which the expressions are singular.

Other examples can be brought to this case by functional manipulations [BFFLS]. For instance the Casimir element C of $\mathfrak{so}(\ell)$ representing *angular momentum*, which can be written $C = p^2q^2 - (pq)^2 - \ell(\ell-1)\frac{\hbar^2}{4}$, has $n(n+(\ell-2))\hbar^2$ for spectrum. For the *hydrogen atom*, with Hamiltonian $H=\frac{1}{2}p^2-|q|^{-1}$, the Moyal product on $\mathbb{R}^{2\ell+2}$ ($\ell=3$ in the physical case) induces a star product on $X=T^*S^\ell$; the energy levels, solutions of $(H-E)*\phi=0$, are found from (13) and the preceding calculations for angular momentum to be (as they should, with $\ell=3$) $E=\frac{1}{2}(n+1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E\in\mathbb{R}^+$ for the continuous spectrum.

We thus have recovered, in a completely autonomous manner entirely within deformation quantization, the results of “conventional” quantum mechanics in these typical examples (and many more can be treated similarly). It is worth noting that the term $\frac{\ell}{2}$ in the harmonic oscillator spectrum, obvious source of divergences in the infinite-dimensional case, disappears if the normal star product is used instead of Moyal – which is one of the reasons it is preferred in field theory.

II.3.2 Symmetries, path integrals and other remarks. Weyl [We31] has, almost from the beginning of quantum theory, stressed the importance, there also, of symmetries, acting on physical states by their unitary irreducible representations (UIR). Wigner and Racah (Flato’s adviser), and many more, have shown all the power of that tool. The former’s contribution in that domain has been widely recognized, including in 1963 by the Nobel Committee for “(...) the discovery and application of fundamental symmetry principles”. He is even the object of a personality cult by a number of fans (see in particular `http://www.physics.umd.edu/robot/wigner/wigma.html`). The impact of Racah, who disappeared prematurely in 1966, is now largely underscored – though he was a virtuose of theoretical spectroscopy and knew very well its power and limitations. So did Flato, especially with respect to particle theory, and he expressed it in a way which that time seemed heretic to many.

The first example of a “complete” classification of UIR for a noncompact real Lie group (modulo some gaps that were later filled by Gårding, the existence of a dense space of differentiable vectors, and Bargmann, the “little group” $SL(2,\mathbb{R})$) is that of the UIR of the Poincaré group $SO(3,1)\cdot\mathbb{R}^4$, by Wigner in 1939 (a problem posed by his brother in law Dirac). The finite dimensional UIR for the compact and complex simple cases go back essentially to Cartan and Weyl’s "unitary trick". Harish Chandra, whose impact of harmonic analysis on semi-simple Lie groups has been prominent, started as a mathematical physicist studying covariant differential equations. Symmetries and quantum theory are thus naturally interrelated. We find the same situation in our context.

Invariance and covariance of star products. The Poisson bracket P is (by definition) invariant under all symplectomorphisms, i.e. transformations of the manifold X generated by the flows $x_u = i(\Lambda)(du)$ defined by Hamiltonians $u \in N$. But already on $\mathbb{R}^{2\ell}$ one sees easily that its powers P^r, hence also the Moyal bracket (3), are invariant only under flows generated by Hamiltonians u which are polynomials of maximal order 2, forming the "affine symplectic" Lie algebra $\mathfrak{sp}(\mathbb{R}^{2\ell}) \cdot \mathfrak{h}_\ell$. For other orderings the invariance is even smaller (only $\mathfrak{h}_\ell$ remains). For general Vey products the first terms of a star product are [BFFLS] $C_2 = P_\Gamma^2 + bH$ and $C_3 = S_\Gamma^3 + T + 3\partial H$. Here H is a differential operator of maximal order 2, T a 2-tensor corresponding to a closed 2-form, ∂ the Chevalley coboundary operator. P_Γ^2 is given (in canonical coordinates) by an expression similar to P^2 in which usual derivatives are replaced by covariant derivatives with respect to a given symplectic connection Γ (a torsionless connection with totally skew-symmetric components when all indices are lowered using Λ). S_Γ^3 is a very special cochain given by an expression similar to P^3 in which the derivatives are replaced by the relevant components of the Lie derivative of Γ in the direction of the vector field associated to the function (u_1 or u_2). Fedosov's algorithmic construction [Fe94] shows that the symplectic connection Γ plays a role at all orders. Therefore the invariance group of a star product is a subgroup of the *finite-dimensional* group of symplectomorphisms preserving a connection. Its Lie algebra is $\mathfrak{g}_0 = \{a \in N; [a,u] = P(a,u)\, \forall u \in N\}$, the elements of which are called *preferred observables*, i.e., Hamiltonians for which the classical and quantum evolutions coincide. We are thus lead to look for a weaker notion and shall call a star product *covariant* under a Lie algebra $\mathfrak{g}$ of functions if the deformed bracket $[a,b] = P(a,b)\ \forall a,b \in \mathfrak{g}$. It can be shown [ACMP] that $*$ is $\mathfrak{g}$-covariant iff there exists a representation τ of the Lie group G whose Lie algebra is $\mathfrak{g}$ into $\mathrm{Aut}(N[[\nu]];*)$ such that $\tau_g u = (Id_N + \sum_{r=1}^{\infty} \nu^r \tau_g^r)(g.u)$ where $g \in G, u \in N$, G acts on N by the natural action induced by the vector fields associated with $\mathfrak{g}$, $(g \cdot u)(x) = u(g^{-1}x)$, and where the τ_g^r are differential operators on W. Invariance of course means that the geometric action preserves the star product: $g \cdot u_1 * g \cdot u_2 = g \cdot (u_1 * u_2)$. This is the basis for the mathematical theory of star representations that we shall briefly present now, which underlines the relevance of all aspects of star products for important problems in group theory and was examplified including in recent works (see e.g. [ADS02]) based on the Kontsevich approach.

Star representations. Let G be a Lie group (connected and simply connected), acting by symplectomorphisms on a symplectic manifold X (e.g. coadjoint orbits in the dual of the Lie algebra $\mathfrak{g}$ of G). The elements $x, y \in \mathfrak{g}$ will be supposed to be realized by functions u_x, u_y in N so that their Lie bracket $[x,y]_{\mathfrak{g}}$ is realized by $P(u_x, u_y)$. Now take a G-covariant star product $*$, that is $P(u_x,u_y) = [u_x,u_y] \equiv (u_x * u_y - u_y * u_x)/2\nu$, then the map $\mathfrak{g} \ni x \mapsto (2\nu)^{-1} u_x \in N$ is a Lie algebra morphism. We can now define the *star exponential*

$$E(e^x) = \mathrm{Exp}(x) \equiv \sum_{n=0}^{\infty} (n!)^{-1} (u_x/2\nu)^{*n} \tag{14}$$

where $x \in \mathfrak{g}$, $e^x \in G$ and the power $*n$ denotes the n^{th} star power of the corresponding function. The

map E extends naturally to a *group homomorphism* $E : G \longrightarrow (N[[\nu,\nu^{-1}]],*)$ where, in the formal series, ν and ν^{-1} are treated as independent parameters for the time being. Alternatively, the values of E can be taken in the algebra $(\mathscr{P}[[\nu^{-1}]],\star)$, where $\mathscr{P}$ is the algebra generated by $\mathfrak{g}$ with the $*$-product (it is a representation of the enveloping algebra).
A *star representation* [BFFLS] of G is a distribution $\mathscr{E}$ (valued in ImE) on X defined by

$$D \ni f \mapsto \mathscr{E}(f) = \int_G f(g)E(g^{-1})dg$$

where D is some space of test-functions on G. The corresponding *character* χ is the (scalar-valued) distribution defined by $D \ni f \mapsto \chi(f) = \int_X \mathscr{E}(f)d\mu$, $d\mu$ being a quasi-invariant measure on X.

The character is one of the tools which permit a comparison with usual representation theory. For semi-simple groups it is singular at the origin in irreducible representations, which may require caution in computing the star exponential (14). In the case of the harmonic oscillator that difficulty was masked by the fact that the corresponding (harmonic oscillator) representation of $\mathfrak{sl}(2)$ generated by (p^2,q^2,pq) is integrable to a double covering of $\mathrm{SL}(2,\mathbb{R})$ and decomposes into a sum $D(\frac{1}{4}) \oplus D(\frac{3}{4})$: the singularities at the origin cancel each other for the two components.

This theory is now very developed, and parallels in many ways the usual (operatorial) representation theory. A detailed account of all the results would take us too far, but among the most notable one may quote an exhaustive *autonomous* treatment of *nilpotent* and even *solvable* Lie groups [AC]. The coadjoint orbits are there symplectomorphic to $\mathbb{R}^{2\ell}$. Polarizations are not required, and "star polarizations" can always be introduced to compare with usual theory. Wavelets, important in signal analysis, are manifestations of star products on the (2-dimensional solvable) affine group of $\mathbb{R}$ or on a similar 3-dimensional solvable group [BB98]. For *semi-simple* Lie groups an array of results is already available, including a complete treatment of the *holomorphic discrete series* and invariant star products on Kähler and Hermitian symmetric spaces (convergent for an appropriate dense subalgebra). Note however that for semi-simple groups the coadjoint orbits alone are no more sufficient for the unitary dual and one needs far more elaborate constructions. For semi-direct products, and in particular for the Poincaré and Euclidean groups, an autonomous theory has also been developed. Comparison with the usual results of "operatorial" theory of Lie group representations can be performed in a number of ways, in particular by constructing an invariant Weyl transform generalizing (1), finding "star polarizations" that always exist (in contradistinction with the geometric quantization approach where in most cases one can at best find complex polarizations), study of spectra (of elements in the center of the enveloping algebra and of compact generators) in the sense considered in II.3.1, comparison of characters, etc. We stress again that the theory of star representations is *autonomous* and can be formulated completely within this framework, based on coadjoint orbits (and additional ingredients when required).

Quantum groups [Dr86, Ji85, SS93, ES98] can be viewed as an avatar of deformation quantization when the category in question is that of Hopf algebras, in this case an algebra N of functions on a Poisson-Lie group G or a topologically dual algebra N', "completed" enveloping algebra of a Lie bialgebra. The topological theory is well established in the semi-simple case [BFGP]), where one has preferred deformations (only the product, or the coproduct, are deformed), but the approach is general. We shall not develop further this aspect here, referring to the original papers and references quoted in recent reviews such as [FS94, St98].

Path integrals, field theory. Path integrals are intimately connected to star exponentials. In fact, in quantum mechanics the path integral of the action is nothing but the partial Fourier transform of the star exponential (11) with respect to the momentum variables, for $X = \mathbb{R}^{2\ell}$ as phase space with the Moyal star product. For normal ordering the path integral is essentially the star exponential [Dit]. For compact groups the star exponential E can be expressed in terms of unitary characters; the star exponential of any Hamiltonian on G/T (where T is a maximal torus in the compact group G) is then equal to the path integral for this Hamiltonian.

The deformation quantization of a given classical field theory consists in giving a proper definition for a star product on the infinite-dimensional manifold of initial data for the classical field equation and constructing with it, as rigorously as possible, whatever physical expressions are needed. As in other approaches to field theory, here also one faces serious divergence difficulties as soon as one is considering interacting fields theory, and even at the free field level if one wants a mathematically rigorous theory. But the philosophy in dealing with the divergences is significantly different and one is in position to take advantage of the cohomological features of deformation theory to perform what can be called *cohomological renormalization*.

Consider a (classical) free massive scalar field Φ with initial data (ϕ, π) in the Schwartz space $\mathscr{S}$. The initial data (ϕ, π) can advantageously be replaced by their Fourier modes $(\bar{a}, a)$, which for a massive theory are also in $\mathscr{S}$ seen as a real vector space. After quantization $(\bar{a}, a)$ become the usual creation and annihilation operators, respectively. The normal star product $*_N$ can then be expressed, in integral representation, by [Dit]:

$$(F *_N G)(\bar{a}, a) = \int_{\mathscr{S}' \oplus \mathscr{S}'} d\mu(\bar{\xi}, \xi) F(\bar{a}, a + \xi) G(\bar{a} + \bar{\xi}, a), \tag{15}$$

where μ is the unique Gaussian measure admitting $\exp(-\frac{1}{4}(\bar{a}|a))$ as characteristic function and F, G are holomorphic functions of $(\bar{a}, a)$ with semi-regular kernels.

Starting with some star product $*$ (e.g. a star product similar to the normal star product (15)) on the manifold of initial data, one would interpret various divergences appearing in the theory in terms of coboundaries (or cocycles) for the relevant Hochschild cohomology. Suppose that we are suspecting that a term in a cochain of the product $*$ is responsible for the appearance of divergences: We can try to eliminate it, or at least get a lesser divergence, by subtracting at the relevant order a coboundary; we would then get a better theory with a new star product, equivalent to the original one. Furthermore, since in this case we can expect to have at each order an infinity of non equivalent star products, we can try to subtract a cocycle and then pass to a non equivalent star product, the lower order cochains of which are identical to those of the original one. Along the way one should preserve the usual properties of a quantum field theory (Poincaré covariance, locality, etc.) and the construction of adapted star products should be done accordingly. The complete implementation of this program should lead to a cohomological approach to renormalization theory. The interested reader will find in [Dit] calculations performing some steps in the above direction, for free scalar fields and the Klein-Gordon equation, and an example of cancellation of some infinities in $\lambda\phi_2^4$-theory via a λ-dependent star product equivalent to a normal star product. It should be possible (but difficult) to formulate the Connes–Kreimer [CoKr] rigorous renormalization procedure in a way that fits in this pattern.

Kontsevich [Ko97] suggested that his formula comes from a perturbation theory for a related bidimensional topological field theory. The physical origin of the weights and graphs has been elucidated by Cattaneo and Felder [CF00]. They constructed a topological field theory on a disc (the so-called Poisson sigma model) whose perturbation series makes Kontsevich graphs and weights appear explicitly after a finite renormalization. They obtain a path integral formula (with zero Hamiltonian) for the star product, the proof of which is given by the quantization of this Poisson sigma model (a delicate question), performed in the Batalin–Vilkovitsky framework. The perturbative expansion of this theory provides (essentially) the graphs and weights of Kontsevich (the angle function appears as a propagator). This approach also provided a simple way to understand the globalization of Kontsevich star products in more geometrical terms, much along the lines of the Fedosov construction.

Further remarks on convergence and the physical meaning of deformation quantization. We have always considered star products as formal series and looked for convergence only in specific examples, generally in the sense of distributions. The same applies to star exponentials, as long as each coefficient in the formal series is well defined. In the case of the harmonic oscillator or more generally the preferred observables H in Weyl ordering, the study is facilitated by the fact that the powers $(H*)^n$ are polynomials in H.

Integral formulas for star products and star exponentials can be used to give a meaning to these expressions beyond the domains where the sums are convergent or summable. *The sums must not be considered as perturbation series*, rather taken as a whole, whether expressed as integral formulas or formal series.

We stress that deformation quantization should not be seen as a mere reformulation of quantum mechanics or quantum theories in general. At the conceptual level, it is the true mathematical formulation of physical reality whenever quantum effects have to be taken into account. The above examples show that one can indeed perform important quantum mechanical calculations, in an autonomous manner, entirely within deformation quantization – and get the results obtained in conventional quantum mechanics. Whether one uses an operatorial formulation or some form of deformation quantization formulation is thus basically a practical question, which formulation is the most effective, at least in the cases where a Weyl or Wigner map exists. When such a map does not exist, a satisfactory operatorial formulation will be very hard to find, except locally on curved phase space, and deformation quantization is the solution.

One can of course (and do in practical examples, also for algebraic varieties [Ko01]) look for small domains (in N) where one has some convergence. We can then speak of “strict” and “semi-formal” deformation quantization. In particular we can look for domains where pointwise convergence can be proved; this was done e.g. for Hermitian symmetric spaces [CGR95]. But it should be clearly understood that one can consider wider classes of observables – in fact, the latter tend to be physically more interesting – than those that fit in a strict C^*-algebraic approach.

III. Composite elementary particles in AdS microworld

It follows from our deformation philospohy that, in order to *anticipate* new formalisms, we have to study deformations of the algebraic structures attached to a given formalism. The only question is, in which category we perform this search for deformations. Usually physics is rather conservative and if we start e.g. with the category of associative or Lie algebras, we tend to deform in this category. In the passage from Galilean physics to special relativity (new parameter c^{-1}, where c is the speed of light), we deform the symmetry of the theory. In the Lie group (or algebra) category, there is a further deformation, giving rise to physics in (anti) de Sitter space-time (the new parameter being the curvature). It is this last aspect which we shall present here.

Recent experimental data indicate that the cosmological constant is most likely positive, suggesting (assuming a space-time of constant curvature, at least in first approximation) a de Sitter universe at cosmological distances. At our level, for (almost) all practical purposes, space-time is Minkowskian (flat). We shall assume that, at a much smaller scale, a tiny constant negative curvature is present, i.e. an anti de Sitter (AdS) microworld. As we shall see, that hypothesis has far reaching consequences that among others could permit to go beyond the Standard Model and in particular explain neutrino oscillations and *PC* violation, and predict new mesons associated with multiple Higgs.

In the Lie group category the deformation chain stops at AdS but one can deform further the symmetry in the Hopf algebra category, to a quantum AdS, where new and interesting features appear, including some very surprising at root of unity [FHT]. We shall here be conservative and remain within usual AdS. But one should be open to the possibility to deform further, quantizing the symmetry group of AdS space-time and/or even space-time itself, in a "stringy" way or with noncommutative geometry as in [CDV01].

III.1 A qualitative overview. The strategy is the following. AdS_4 group representation theory shows us that the UIR which, for many good reasons (see e.g. [AFFS]), should be called massless, are (in contradistinction with the flat space limit) composed of two more degenerate UIR of (the covering of) the AdS_4 group $SO(3,2)$. The latter were discovered by Dirac [Di63] and called singletons because the states appear on a single line and not on a lattice. They are naturally confined because their energy is proportional to momentum times the tiny curvature, which would require a laboratory of cosmic dimensions to get a measurable energy. We have called them *Di* and *Rac*, on the pattern of "bra" and "ket", and because the *Di* has 2 components – helicity $\frac{1}{2}$ – and the *Rac* only one ('only' is translated by 'rac' in Hebrew). They are the massless representations of the Poincaré group in 2+1 dimensional space, where $SO(3,2)$ is the conformal group (AdS_4/CFT_3 correspondence). In the flat space limit the singletons reduce to vacua, representations of the Lorentz group trivial on the Poincaré translations. One of the reasons we call the massless UIR of $SO(3,2)$ massless is that they can be (uniquely, up to the helicity sign which is absent in AdS where time is compact) extended to the conformal group $SO(4,2)$ (the same for AdS and Minkowski) and restricted to Poincaré to the Wigner massless UIR – and vice-versa. Another one is the presence of gauges (see below) and a third is the contraction from $SO(3,2)$ to Poincaré at the level of these representations. $Di \oplus Rac$ itself happens to be the tensor square of the harmonic oscillator representation of (the covering of) $SO(2,1)$, i.e. a representation of (the covering of) $SO(2,2)$, the linear part of the infinite-dimensional conformal group of 1+1 dimensional space-time (again, some AdS_3/CFT_2 correspondence and a possible dimensional reduction in the theory to be developed).

So far that compositeness is kinematical. Dynamics require in particular the consideration of field equations, initially at the first quantized level, in particular the analogue of the Klein-Gordon equation in AdS_4 for the *Rac*. There, as can be expected of massless (in 2+1 space) representations, gauges appear. We thus have to deal with indecomposable representations, triple extensions of UIR, as in the Gupta-Bleuler (GB) theory, and their tensor products. It is also desirable to take into account conformal covariance at these GB-triplets level. The situation gets therefore much more involved, quite different from the flat space limit, which makes the theory even more interesting.

One can then attempt to "plug into" conventional QED by considering a massless photon composed of two scalar singletons. The idea is to take creation and annihilation operators for the *Rac* that satisfy unusual commutation relations (which is fine for confined entities) in such a way that for the 2-*Rac* states (photons), the creation and annihilation operators satisfy the usual canonical commutation relations (CCR). We thus get a new and interesting infinite-dimensional Lie algebra, a kind of "square root" of the CCR. The theory can be completed, including taking into account conformal covariance of triplets, and composite QED was established [FF88].

After QED the natural step is to introduce compositeness in electroweak theory. Along the lines described above, that would require finding a kind of "square root of superalgebra", with both CAR and CCR included, obtained from creation and annihilation operators for $Di \oplus Rac$. That has yet to be done. Some steps in that direction have been initiated but the mathematical problems are extremely complicated, even more so since now the three flavors of leptons have to be considered.

But here a more pragmatic approach can be envisaged [Fr00], triggered by recent experimental data which indicate that there are oscillations between various flavors of neutrinos. The latter would thus not be massless. This is not as surprising as it seems from the AdS point of view, because one of the attributes of masslessness is the presence of gauges. These are group theoretically associated with the limit of unitarity in the representations diagram, and the neutrino is above that limit in AdS: the *Di* is at the limit. Thus, all 9 leptons can be treated on an equal footing. One is then tempted to write them in a square table and consider them as composites $L^A_\beta = R^A D_\beta$. (We know, but do not necessarily tell phenomenologists in order not to scatter them away with a high brow theory, that they are $Di - Rac$ composites.) In this empirical approach, the vector mesons of the electroweak model are $Rac - Rac$ composites and the model predicts a new set of vector mesons that are $Di - Di$ composites and that play exactly the same role for the flavor symmetry $U_F(2)$ as the weak vector bosons do for the weak group $U_W(2)$. A set of (maybe five pairs of) Higgs fields would have Yukawa couplings to the leptons currents and massify the leptons (and the vector mesons and the new mesons). This attempt has been developed in part in [Fr00] (Frønsdal and I are still pursuing that direction) and is qualitatively promising. In addition to the neutrino masses it could explain why the Higgs has so far escaped detection: instead of one "potato" on has a gross puree of five, far more difficult to isolate from background. Quantitatively however its predictive power is limited by the presence of too many free parameters. Maybe the addition to the picture of a deformation induced by the strong force and of the 18 quarks, which could be written in a cube and also considered composite (of 3 constituents when the strong force is introduced), would make this "composite standard model" more predictive.

III.2 A brief more precise overview of singleton symmetry and field theory. We have, since about 25 years, developed many aspects of the theory, of which a qualitative survey has been presented in the first half of the preceding subsection. References and a short account can be found in [FFS99] of which we shall now, so as to give more background to the previous discussion, present some highlights. We consider mainly the twofold covering $SO_{(2,3)}$ of the connected component of the identity of $SO(2,3)$, and denote by $D(E_0,s)$ its minimal weight representations. Here E_0 is the minimal $SO(2)$ eigenvalue and the half-integer s is the spin. These irreducible representations are unitary (belonging to the discrete series at or above the limit of unitarity) provided $E_0 \geq s+1$ for $s \geq 1$ and $E_0 \geq s+\frac{1}{2}$ for $s=0$ and $s=\frac{1}{2}$. At the limit of unitarity (i.e. when $2E_0$, which is an integer for $SO_{(2,3)}$ but can take any value for the universal covering, tends to the limit from above), the Harish Chandra module $D(E_0,s)$ becomes indecomposable and the physical UIR appears as a quotient, a hall-mark of gauge theories. For instance, for $s \geq 1$, we get in the limit an indecomposable representation $ID(s)$: $D(s+1,s) \rightarrow D(s+2,s-1)$, a shorthand notation [FF88] for what mathematicians would write as a short exact sequence of modules $0 \rightarrow D(s+1,s) \rightarrow ID(s) \rightarrow D(s+2,s-1) \rightarrow 0$.

In gauge theories one needs extensions involving more than two UIRs. A typical situation is the case of flat space electromagnetism where one has the classical Gupta-Bleuler triplet which, in our shorthand notations, can be written $Sc \rightarrow Ph \rightarrow Ga$. Here Sc (scalar modes) and Ga (gauge modes) are massless zero-helicity UIRs $h(0,0)$ of the Poincaré (inhomogeneous Lorentz) group $\mathscr{P}_{1+3} = SO_{(1,3)} \cdot \mathbb{R}^4$ while Ph is the module of physical modes, transforming under $h(0,1) \oplus h(0,-1)$, where $h(0,s)$ is the UIR of $\mathscr{P}_{1+3}$ with mass 0 and helicity $s \in \mathbb{Z}$. The scalar modes can be suppressed by a gauge fixing condition (e.g. the Lorentz condition) but then one is left with a nontrivial extension $Ph \rightarrow Ga$ on the vector space $Ph \dotplus Ga$ which has no invariant nondegenerate metric and cannot be quantized covariantly. However the above Gupta-Bleuler triplet, a nontrivial successive extension $Sc \rightarrow (Ph \rightarrow Ga)$, is an indecomposable representation on a space which admits an invariant nondegenerate (but indefinite) Hermitian form and it must be used in order to obtain a covariant quantization of this gauge theory. We shall meet here a similar situation, which in fact cannot be avoided. Indeed a general result says that if an extension $U^1 \rightarrow U^2$, with U^2 a UIR, has a nondegenerate Hermitian form, then U^1 is equivalent to an extension $U^2 \rightarrow U^3$ for some representation U^3 and the original extension is in fact a triplet $U^2 \rightarrow U^3 \rightarrow U^2$. The *massless representations* of $SO(2,3)$ are defined (for $s \geq \frac{1}{2}$) as $D(s+1,s)$ and (for helicity zero) $D(1,0) \oplus D(2,0)$.

For $s=0$ and $s=\frac{1}{2}$, the above mentioned gauge theory appears not at the level of the massless representations $D(1,0) \oplus D(2,0)$ and $D(\frac{3}{2},\frac{1}{2})$ but at the limit of unitarity, the singletons $Rac = D(\frac{1}{2},0)$ and $Di = D(1,\frac{1}{2})$. These UIRs remain irreducible on the Lorentz subgroup SO$(1,3)$ and on the (1+2) dimensional Poincaré group $\mathscr{P}_{1+2}$, of which $SO_{(2,3)}$ is the conformal group. On $\mathscr{P}_{1+2}$ they give the only massless (discrete helicity) representations. The singleton representations have a fundamental property: $(Di \oplus Rac) \otimes (Di \oplus Rac) = (D(1,0) \oplus D(2,0)) \oplus 2\bigoplus_{s=\frac{1}{2}}^{\infty} D(s+1,s)$. Note that all the representations that appear in the decomposition are massless representations. Thus, in contradistinction with flat space, in AdS$_4$, massless states are “composed” of two singletons. The flat space limit of a singleton is a vacuum (a representation of $\mathscr{P}_{1+3}$ which is trivial on the translations) and, even in AdS$_4$, the singletons are very poor in states: their (E,j) diagram has only a single trajectory (hence their name). In normal units a singleton with angular momentum j has energy $E = (j+\frac{1}{2})\rho$, where ρ is the curvature of the AdS$_4$ universe. This means that only a laboratory of cosmic dimensions can detect a j large enough for E to be measurable. Elementary particles would then be composed of two, three or more singletons and/or anti singletons (the latter being associated with the contragredient representations). As with quarks, several (possibly three, so far) flavors of singletons (and anti singletons) should eventually be introduced to account for all elementary particles. In order to pursue this point further we need to give a little more details on how to develop a field theory of singletons and of particles composed of singletons.

For reasons explained in [FFd87, FFS99] and references quoted therein, we consider for the *Rac*, the dipole equation $(\Box - \frac{5}{4}\rho)^2\phi = 0$ with the boundary conditions $r^{\frac{1}{2}}\phi < \infty$ as $r \to \infty$, which carries the non-decomposable representation $D(\frac{1}{2},0) \to D(\frac{5}{2},0)$. Quantization needs a non-degenerate, invariant symplectic structure. This requires the introduction of additional modes, canonically conjugate to the gauge modes (compare the situation in electrodynamics where Maxwell theory has no momentum conjugate to gauge modes), to give to the total space the symmetric form $D(\frac{5}{2},0) \to D(\frac{1}{2},0) \to D(\frac{5}{2},0)$ or " scalar $\to$ transverse $\to$ gauge." A remarkable fact is that this theory is a *topological field theory*; that is [FF81], the physical solutions manifest themselves only by their boundary values at $r \to \infty$: $\lim r^{\frac{1}{2}}\phi$ defines a field on the 3-dimensional boundary at infinity. There, on the boundary, gauge invariant interactions are possible and make a 3-dimensional conformal field theory (CFT). A 5-dimensional analogue of this 4-dimensional theory is the 5-dimensional Anti de Sitter/4-dimensional conformal field theory (AdS_5/CFT_4) duality which has found an interesting interpretation by Maldacena [Ma97] in the context of strings and branes.

However, if massless fields (in 4 dimensions) are singleton composites, then singletons must come to life as four dimensional objects, and this requires the introduction of unconventional statistics. The requirement that the bilinears have the properties of ordinary (massless) bosons also tells us that the statistics of singletons must be of another sort. The basic idea is [FF88] that we can decompose the singleton field operator as $\phi(x) = \sum_{-\infty}^{\infty} \phi^j(x) a_j$ in terms of positive energy creation operators $a^{*j} = a_{-j}$ and annihilation operators a_j (with $j > 0$) without so far making any assumptions about their commutation relations. The choice of commutation relations comes later, when requiring that photons, considered as $2 - Rac$ fields (using the full tensor product of the two singleton triplets) be Bose-Einstein quanta. The singletons are then subject to unconventional statistics (which is perfectly admissible since they are naturally confined) and an appropriate Fock space can be constructed. Based on these principles, a (conformally covariant) composite QED theory was constructed [FF88], with all the good features of the usual theory. In addition the BRST structure of singleton gauge theory induces [FF87] the BRST structure of the electromagnetic potential.

A more recent contribution [FF98] to this interpretation of massless fields as singleton composites deals with gravitons, giving an explicit expression for the weak gravitational potential in terms of singleton bilinears. If this idea is introduced in the context of bulk/boundary duality, it is natural to relate massless fields on the bulk to conserved currents on the boundary. But we are interested in the composite nature of massless fields on space time (the bulk), and a direct current-field identity is then inappropriate, since currents are conserved by virtue of the field equations while massless fields are divergenceless only on the physical subspace defined by gauge fixing. It was shown [FF98] that the dipole formulation provides a natural construction of all massless fields in terms of bilinears that are conserved only by virtue of the gauge fixing condition on constituent singleton fields.

IV. Nonlinear covariant field equations

We shall begin with a short summary of the theory of nonlinear Lie group representations [FPS77], started by our group about 25 years ago, and continue with a very short presentation, based on [FST97], of recent spectacular applications, to Nonlinear Klein-Gordon equations and especially Maxwell-Dirac equations (first-quantized electrodynamics). The references for this Section can be found in [FSST, FST97]. In such equations the nonlinearity appears usually as coupled to the linear (free) equations, with a coupling constant that plays the role of deformation parameter. Once the classical covariant field equations are studied enough in details one can think [Dit] of studying their quantization along the lines described in Section II. This part is thus the natural third component of our trilogy.

IV.1 NonLinear Representations and Cohomology. Let E be a Banach or Fréchet space. Denote by $\mathscr{L}_n(E)$ the space of *continuous symmetric multilinear* maps $E^n \to E$. It is isomorphic to the space $\mathscr{L}(\hat{\otimes}^n E, E)$ of linear continuous maps from the completed (with respect to the projective tensor product topology) symmetric tensor product of n copies of E into E. To any f^n in $\mathscr{L}_n(E)$ one associates the monomial $\hat{f}^n$: $E \to E$ by $\hat{f}^n(\varphi) \equiv f^n(\varphi,\dots,\varphi)$, $\varphi \in E$. We denote by $\mathscr{F}(E)$ the space of *formal series* $f = \sum_{n=1}^\infty f^n$ with product $\bullet$ given by the composition of the corresponding maps $\sum_{n=1}^\infty \hat{f}^n$ from E to E, i.e. if $h = \sum_1^\infty h^n \in \mathscr{F}(E)$ then the n-th term of the product is such that $E \ni \widehat{f \bullet h}^n(\varphi) = \sum_{1\le p\le n} f^p(\sum_{i_1+\dots+i_p=n} \hat{h}^{i_1}(\varphi)\otimes\dots\otimes\hat{h}^{i_p}(\varphi))$. A (formal) *nonlinear representation* of a Lie group G is a couple (S,E) where S is a homomorphism $G \ni g \mapsto S_g = \sum_{n=1}^\infty S_g^n$ of G into the group of *invertible* elements of $\mathscr{F}(E)$. Similarly one defines the notion of *analytic* nonlinear representation by assuming that the morphism S is analytic at 0 (when $\mathscr{F}(E)$ is endowed with a natural topology). The *equivalence* of nonlinear representations is defined by an invertible intertwining operator $A \in \mathscr{F}(E)$: $(S,E) \sim (S',E) \Leftrightarrow S'_g = AS_gA^{-1}$, and S is said *linearizable* if $S \sim S^1$, S^1 being its linear part.

Take $\tilde{E} = \cup_n \oplus_{p=1}^n (\hat{\otimes}^p E)$ ("Fock space without vacuum"). A one-to-one algebra homomorphism $\Lambda : \mathscr{F}(E) \to \mathscr{L}(\tilde{E})$ mapping $\bullet$ to the product of linear maps is defined on the n-th level by $\Lambda(f)^n = \sum_{1\le p\le n}\sum_{i_1+\dots+i_p=n}(f^{i_1}\otimes\dots\otimes f^{i_p})$. Its differential $d\Lambda$ is then defined on the n-th level by $d\Lambda(f)^n = \sum_{1\le p\le n}(\sum_{0\le q\le p-1} I_q \otimes f^{n-p+1}\otimes I_{p-q-1})$ and maps the bracket $f\times h - h\times f$ of formal series (which is therefore a Lie bracket) into the commutator of the linear operators $d\Lambda(f)$ and $d\Lambda(h)$; $f\times h$ is defined on the n-th level (denoting by $\circ$ the composition of maps) by the "differentiated" formula $(f\times h)^n = \sum_{1\le p\le n} f^p \circ (\sum_{0\le q\le p-1} I_q \otimes h^{n-p+1}\otimes I_{p-q-1})$. A *nonlinear representation of a Lie algebra* $\mathfrak{g}$ is a Lie morphism of $\mathfrak{g}$ into $\mathscr{F}(E)$ equipped with this bracket. An important result of the theory is that one has a *Lie theory*: on the Fréchet space E_∞ of *differentiable vectors of the linear part* S^1, that can be realized as the linear span of the vectors of the form $\{\varphi = \int_G S_g^1 \psi\alpha(g)dg\}$, with $\psi \in E$ and $\alpha \in \mathscr{D}(G)$ (C^∞ functions on G with compact support), to any nonlinear representation S of G one can associate its differential dS, a nonlinear representation of $\mathfrak{g}$, *and vice-versa*. The result holds in both the formal and analytic frameworks. For a large class of nonlinear analytic Lie algebra representations dS (including those arising from physical nonlinear evolution equations), the space of differentiable vectors for S coincides with the space of differentiable vectors for its linear part S^1. Note that if λ is a scalar, $f(\lambda\varphi) = \sum_{n=1}^\infty \lambda^n \hat{f}^n(\varphi)$ and we have a formula very similar to those appearing in deformation theory: A nonlinear representation is a perturbation of a linear one by nonlinear terms.

Let G be a group, U a linear representation of G in E, $g,g' \in G$. *1-cocycles* are maps $R: G \to E$ such that $R_{gg'} = R_g + U_g R_{g'}$. We denote their space by $Z^1(G,E)$. *1-coboundaries* are 1-cocycles for which there exists a (0-cochain) $\varphi \in E$ such that $R_g = U_g\varphi - \varphi$ and we denote their space by $B^1(G,E)$. The ("affine") 1-cohomology is the quotient $H^1(G,E) = Z^1(G,E)/B^1(G,E)$. An *extension* (V,F) of (U,E) by (U',E') is an exact sequence of G-modules $0 \to E \to F \to E' \to 0$ and is given by a *1-cocycle* of G for the representation T in $\mathscr{L}(E',E)$ defined by $\mathscr{L}(E',E) \ni A \mapsto T_g A = U_g A U'_{g^{-1}} \in \mathscr{L}(E',E)$. The representation is equivalent to a direct sum if this cocycle is a coboundary.

Nonlinear representations as successive extensions. Let (S,E) be a nonlinear representation of G, $S = \sum_{n=1}^{\infty} S^n$. The group law gives: $S^2_{gg'} = S^1_g \circ S^2_{g'} + S^2_g \circ (S^1_{g'} \otimes S^1_{g'})$, i.e. $R_g \equiv S^1_g \circ S^2_{g^{-1}}$ is a 1-cocycle of G in $\mathscr{L}(\hat{\otimes}^2 E,E)$ for the representation T given by: $\mathscr{L}(\hat{\otimes}^2 E,E) \ni A \mapsto T_g A = S^1_g \circ A \circ (S^1_{g^{-1}} \otimes S^1_{g^{-1}})$. Therefore on $\tilde{E}$, S is obtained by *successive extensions* of S^1 by its (symmetric) tensorial powers $\otimes^n S^1$, $n \geq 2$: first S^1 by $S^1 \otimes S^1$, then the result by $\otimes^3 S^1$ and so on. By standard arguments, if (S,E) is a nonlinear representation of a Lie group G in a Fréchet space of differentiable vectors, and if the (differentiable) cohomology spaces $H^1(G,\mathscr{L}(\hat{\otimes}^n E,E)) = 0$ $\forall n \geq 2$, S is linearizable. A necessary condition for the 1-cohomology with values in an irreducible representation to be nonzero is that the center of the enveloping algebra be trivially represented. In the general case it is sufficient to have at least one invertible operator in the representation of the center of the enveloping algebra for the corresponding 1-cohomology to vanish.

IV.2 NonLinear Klein-Gordon equation. The nonlinear Klein-Gordon equation (NLKG) can be written as: $(\Box + m^2)\ \varphi(t,x) = P(\varphi(t,x), \frac{\partial}{\partial t}\varphi(t,x), \nabla\varphi(t,x))$ where $m^2 > 0$, $x \in \mathbb{R}^n$, $n \geq 2$ and P is analytic with no constant and no linear term ($P(0) = 0 = dP(0)$). We transform it into an *evolution equation* by setting $\varphi(t)(x) = \varphi(t,x)$, $(\dot{\varphi}(t))(x) = \frac{\partial}{\partial t}\varphi(t,x)$, and defining $f_\varepsilon(t) = \dot{\varphi}(t) + \varepsilon i\delta\varphi(t)$, where $\varepsilon = \pm 1$, $\delta = (m^2 - \Delta)^{1/2}$ and $F(f(t)) = P(\varphi(t),\dot{\varphi}(t),\nabla\varphi(t))$: $\frac{d}{dt}(f_+(t), f_-(t)) = i\delta f_+(t), -f_-(t) + (F(f(t)),F(f(t)))$. In the following we assume first, for simplicity of the exposition, that P is polynomial and covariant under Poincaré transformations.

Associated NonLinear Representation of the Poincaré Lie Algebra $\mathfrak{p}$. The space is $E = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$, where the differentiable vectors space will be $E_\infty = \mathscr{S}(\mathbb{R}^n) \oplus \mathscr{S}(\mathbb{R}^n)$. We denote by Π the usual basis of the Lie algebra $\mathfrak{p}$ and by Π' that of the enveloping algebra $\mathscr{U}(\mathfrak{p})$ defined by the Poincaré-Birkhoff-Witt theorem. The *linear representation* of $\mathfrak{p}$ is defined by: $T^1_{P_0}(f_+,f_-) = i\delta(f_+,-f_-)$; $T^1_{P_j} = \partial_j$, $(1 \leq j \leq n)$; $T^1_{M_{ij}} = (x_i\partial_j - x_j\partial_i)$; $T^1_{M_{0j}}(f_+,f_-) = (i\delta x_j f_+, -i\delta x_j f_-)$. It exponentiates to a linear representation U^1 of the Poincaré group $\mathscr{P}$ on E_∞ which extends to a unitary representation (still denoted by U^1) on E. The *nonlinear representation* of $\mathfrak{p}$ is then defined by $T_X = T^1_X + \tilde{T}_X$, $X \in \mathfrak{p}$ with $T^1_X \in \mathscr{L}(E_\infty)$ and $T_X : E_\infty \to E_\infty$, where $\tilde{T}_{P_0}(f) = (F(f),F(f))$; $\tilde{T}_{P_i}(f) = 0 = \tilde{T}_{M_{ij}}(f)$; $\tilde{T}_{M_{0j}}(f) = (x_j F(f), x_j F(f))$. With these notations the nonlinear Klein-Gordon equation takes the form $\frac{d}{dt}f(t) = (T^1_{P_0} + \tilde{T}_{P_0})f(t)$.

The results. We introduce appropriate Banach spaces which are completions of the *differentiable vectors* space E_∞. Then *local solutions* are obtained by Lie theory [FPS77]. *Global solutions* will follow from the linearizability of the time translations which (together with asymptotic freedom) will be a consequence of the existence of a solution to the related integral Yang-Feldman-

Källén equation. In this way we are able to obtain global nonlinear representations, analytic linearizability, global solutions and asymptotic completeness. For precise statements, see [FSST] and references therein. A similar result holds also (with the obvious modifications) when P is only C^∞. These methods permit to include quadratic interactions in the equation in physical 1+3 dimensions, that had not been treated before. All these are scalar field equations. Equations involving massless particles are more difficult to treat, in particular due to infrared divergencies. Nevertheless, as we shall see in the next subsection, the general framework presented here is powerful enough to permit their treatment.

IV.3 Asymptotic completeness, global existence and the infrared problem for the Maxwell-Dirac equations. This subsection is based on the extensive monograph [FST97] and especially on its introduction where the main results are sketched.

It is well known that the construction of the observables on the Fock space of QED (Quantum Electrodynamics) requires infrared corrections to eliminate the infrared divergencies in the perturbative expression of the quantum scattering operator. These corrections are introduced by hand with the purpose to give a posteriori a finite theory. We shall here present rigorous results which have been obtained concerning the infrared problem for the (classical) Maxwell-Dirac equations. The belief is that such results can *a priori* be of interest for QED, especially for the infrared regime and combined with the deformation quantization approach. The results show in particular that, also in the classical case, one obtains infrared divergencies, if one requires free asymptotic fields as it is needed in QED.

The Equations. We use conventional notations: electron charge $e=1$; Dirac matrices γ^μ, $0\leq\mu\leq 3$; metric tensor $g^{\mu\nu}$, $g^{00}=1$, $g^{ii}=-1$ for $1\leq i\leq 3$ and $g^{\mu\nu}=0$ for $\mu\neq\nu$; $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$; $\partial_\mu=\partial/(\partial x^\mu)$; $\Box=\partial_\mu\partial^\mu$. The classical Maxwell-Dirac (MD) equations read: $\Box A_\mu=\overline{\psi}\gamma_\mu\psi$, $0\leq\mu\leq 3$, $(i\gamma^\mu\partial_\mu+m)\psi=A_\mu\gamma^\mu\psi$, $m>0$, $\partial_\mu A^\mu=0$, where $\overline{\psi}=\psi^+\gamma_0$, ψ^+ is the Hermitian conjugate of the Dirac spinor ψ. We write them also as an evolution equation: $\frac{d}{dt}(A_\mu(t),\dot{A}_\mu(t))=(\dot{A}_\mu(t),\Delta A_\mu(t))+(0,\overline{\psi}(t)\gamma_\mu\psi(t))$, $\frac{d}{dt}\psi(t)=\mathscr{D}\psi(t)-iA_\mu(t)\gamma^0\gamma^\mu\psi(t)$ where $\mathscr{D}=-\sum_{j=1}^3\gamma^0\gamma^j\partial_j+i\gamma^0 m$, $\Delta=\sum_{j=1}^3\partial_j^2$, $t\in\mathbb{R}$ and where $A_\mu(t)$, $\dot{A}_\mu(t)\colon\mathbb{R}^3\to\mathbb{R}$, $\psi(t)\colon\mathbb{R}^3\to\mathbb{C}^4$. The Lorentz gauge condition takes on initial conditions $A_\mu(t_0)$, $\dot{A}_\mu(t_0)$, $0\leq\mu\leq 3$, and $\psi(t_0)$ at $t=t_0$ the form (with $A^\mu=g^{\mu\nu}A_\nu$): $\dot{A}^0(t_0)+\sum_{i=1}^3\partial_i A^i(t_0)=0$, $\Delta A^0(t_0)+|\psi(t_0)|^2+\sum_{i=1}^3\partial_i\dot{A}^i(t_0)=0$. The initial data are thus subspaces of triples $(f,\dot{f},\alpha)$ in $L^2(\mathbb{R}^3,\mathbb{R}^4)\oplus L^2(\mathbb{R}^3,\mathbb{R}^4)\oplus L^2(\mathbb{R}^3,\mathbb{C}^4)$. Whenever needed, the index M will refer to the Maxwell part $(f,\dot{f})$ and D will refer to the Dirac part α.

Poincaré covariance: linear part and representation spaces. The MD equations are manifestly covariant under the action of the universal covering $\mathscr{P}_0=\mathbb{R}^4\cdot SL(2,\mathbb{C})$ of the Poincaré group. Therefore one can complete the time translation generator to a nonlinear representation of the whole Lie algebra $\mathfrak{p}=\mathbb{R}^4\cdot\mathfrak{sl}(2,\mathbb{C})$ of $\mathscr{P}_0$. We can now introduce the *topological vector spaces of the representations* and write the linear representation of the Poincaré group there, see [FSST].

Nonlinear Poincaré covariance. A nonlinear representation T of $\mathfrak{p}$ (in E_i^ρ, a space of differentiable vectors for the linear representation, $i=1,\ldots,\infty$)) is obtained by the fact that the MD equations are covariant: $T_X=T_X^1+T_X^2$, $X\in\mathfrak{p}$ where T^1 is the preceding linear part and T^2 is obtained from the interaction part of equation. The gauge condition takes on initial data $u=(f,\dot{f},\alpha)$ the form $\dot{f}_0=\sum_{1\leq i\leq 3}\partial_i f_i$, $\Delta f_0=\sum_{1\leq i\leq 3}\partial_i\dot{f}_i-|\alpha|^2$.

The problem to integrate globally the nonlinear Lie algebra representation T consists then in proving the existence of an open neighbourhood of zero in an appropriate space of differentiable vectors and a C^∞ group action such that $\frac{d}{dt}U_{\exp(tX)}(u)\mid_{t=0}= T_X u$ for $X \in \mathfrak{p}$ and $U_g(0) = 0$ for $g \in \mathscr{P}_0$. We extend T from $\mathfrak{p}$ to to the enveloping algebra $U(\mathfrak{p})$ by defining inductively: $T_{\mathbb{I}} = I$, where $\mathbb{I}$ is the identity element in $\mathscr{U}(\mathfrak{p})$, and $T_{YX} = DT_Y.T_X$, $Y \in \mathscr{U}(\mathfrak{p})$, $X \in \mathfrak{p}$ where $(DA.B)(f)$ is the Fréchet derivative of A at the point f in the direction $B(f)$. It maps E^ρ_∞ to E^ρ_∞. Moreover $\frac{d}{dt}T_{Y(t)}(u(t)) = T_{P_0Y(t)}(u(t))$, $Y \in U(\mathfrak{p})$ where $Y(t) = \exp(t\,\mathrm{ad}_{P_0})Y$, $\mathrm{ad}_{P_0}Z = [P_0,Z]$, if $\frac{d}{dt}u(t) = T_{P_0}(u(t))$. We also define $\tilde{T}_Y$, $Y \in U(\mathfrak{p})$ by $T_Y = T^1_Y + \tilde{T}_Y$.

The Infrared Problem. On the classical level the infrared problem consists of determining to which extent the long-range interaction created by the coupling $A^\mu j_\mu$ between the electromagnetic potential A_μ and the current $j_\mu = \overline{\psi}\gamma_\mu\psi$ is an obstruction for the separation, when $|t| \to \infty$, of the nonlinear relativistic system into two asymptotic isolated relativistic systems, one for the electromagnetic potential A_μ and one for the Dirac field ψ. It has been proved in [FST97] that there is such an obstruction, which in particular implies that *asymptotic in and out states do not transform according to a linear representation of the Poincaré group*. This constitutes a serious problem for the second quantization of the asymptotic (in and out going) fields. Indeed the particle interpretation usually requires free relativistic fields, i.e. at least a linear representation of the Poincaré group (U^1 in our case). Therefore we have to introduce nonlinear representations $U^{(-)}$ and $U^{(+)}$ of $\mathscr{P}_0$, which can give the transformation of the asymptotic in and out states under $\mathscr{P}_0$ and permit a particle interpretation.

Two reasons permit to determine the class of asymptotic representations $U^{(\varepsilon)}$, $\varepsilon = \pm$. First, the classical observables, 4-current density, 4-momentum and 4-angular momentum are invariant under gauge transformations. Second, if the evolution equations become linear after a gauge transformation one can use the freedom of gauge in the second quantization of the fields. It is therefore reasonable to postulate that the asymptotic representations $g \mapsto U^{(\varepsilon)}_g$ are linear modulo a nonlinear gauge transformation depending on g and respecting the Lorentz gauge condition.

Moreover to determine the class of admissible modified wave operators it is reasonable to postulate that solutions of the MD equations should converge when $t \to \pm\infty$ in the appropriate space to free solutions (i.e. solutions of the MD equations with vanishing right-hand side) modulo a gauge transformation, not necessarily respecting the Lorentz gauge condition; such transformations are admissible since they leave invariant the observables. In mathematical terms the infrared problem of the MD equations then consists of determining diffeomorphisms (*modified wave operators*) Ω_ε satisfying $U^{(\varepsilon)}_g = \Omega^{-1}_\varepsilon \circ U_g \circ \Omega_\varepsilon$ with $g \in \mathscr{P}_0$, $\varepsilon = \pm$, where the asymptotic representations are C^∞ functions $U^{(\varepsilon)}$. In order to satisfy the two preceding postulates, we impose supplementary conditions on $U^{(\varepsilon)}$ and Ω_ε, which are justified at the end. They use an auxiliary function which is determined by the fact that it is in a certain sense an approximate solution of the Hamilton-Jacobi equation for a relativistic electron in an external electromagnetic potential.

Presentation of the main results. We refer to [FSST] for a precise definition of the spaces $E^{\circ\rho}_N$ and $E^{\circ\rho}_\infty \approx V^\rho_\infty$ of differentiable vectors satisfying some gauge conditions, and of the representations on them; U^M (resp. U^D) refers to their M (resp. D) restrictions, of which U^{M1} (resp. U^{D1}) are the linear parts. Our aim in quoting the Theorems below is to give the flavor of the results that can be obtained on the basis of such techniques, an idea of their difficulty and of the importance of the symmetry based formalism which provides adequate spaces of initial data.

THEOREM 4.1. *$U^{(+)}\colon \mathscr{P}_0 \times E_N^{\circ\rho} \to E_N^{\circ\rho}$ is a continuous nonlinear representation of $\mathscr{P}_0$ in $E_n^{\circ\rho}$ if $\frac{1}{2} < \rho < 1$ and $N \geq 4$. In addition, the function $U^{(+)}\colon \mathscr{P}_0 \times E_\infty^{\circ\rho} \to E_\infty^{\circ\rho}$ is C^∞. Moreover $U^{(+)}$ is not equivalent by a C^2 map to a linear representation on $E_\infty^{\circ\rho}$.*

THEOREM 4.2. *There exist an open neighbourhood $\mathscr{U}_\infty$ (resp. $\mathscr{O}_\infty^{(+)}$) of zero in V_∞^ρ (resp. $E_\infty^{\circ\rho}$, $\frac{1}{2} < \rho < 1$), a diffeomorphism $\Omega_+\colon \mathscr{O}_\infty^{(+)} \to \mathscr{U}_\infty$ and a C^∞ function $U\colon \mathscr{P}_0 \times \mathscr{U}_\infty \to \mathscr{U}_\infty$, defining a nonlinear representation of $\mathscr{P}_0$, such that, for $u = (f, \dot{f}, \alpha) \in \mathscr{O}_\infty^{(+)}$:*

a) $\frac{d}{dt}U_{\exp(tX)}(u) = T_X(U_{\exp(tX)}(u)), \quad X \in \mathfrak{p},\ t \in \mathbb{R},\ u \in \mathscr{U}_\infty$

b) $\Omega_+ \circ U_g^{(+)} = U_g \circ \Omega_+$

c) $\lim_{t\to\infty}\Big(\|U^M_{\exp(tP_0)}(\Omega_+(u)) - U^{M1}_{\exp(tP_0)}(f,\dot{f})\|_M$
$$+\|U^D_{\exp(tP_0)}(\Omega_+(u)) - \textstyle\sum_{\varepsilon=\pm} e^{is_\varepsilon^{(+)}(u,t,-i\partial)}P_\varepsilon(-i\partial)U^{D1}_{\exp(tP_0)}\alpha\|_D\Big) = 0.$$

This theorem solves in particular *the Cauchy problem* for small initial data and proves *asymptotic completeness.* The solution $(A(t,\cdot),\dot{A}(t,\cdot),\psi(t,\cdot)) = U_{\exp(tP_0)}(u)$ of the Cauchy problem satisfies $\sup_{\substack{x\in\mathbb{R}^3\\ t\geq 0}}\Big((1+|x|+t)^{\frac{3}{2}-\rho}|A_\mu(t,x)| + (1+|x|+t)|\partial_\nu A_\mu(t,x)| + (1+|x|+t)^{\frac{3}{2}}|\psi(t,x)|\Big) < \infty.$

Cohomology. These conditions and results have a natural *cohomological interpretation.* A necessary condition for $U^{(+)}$ and Ω_+ to be a solution of the modified wave operator equation is that the formal power series development of $U_g^{(+)}$, U_g and Ω_+ in the initial conditions satisfy cohomological equations. In particular the second order terms $U_g^{(+)2}$, U_g^2 and Ω_+^2 must satisfy $\delta\Omega_+^2 = C^2, \quad \delta R^{(+)2} = 0$, where δ is a coboundary operator defined by the representation $(g,A) \mapsto U_g^1 A^2(\otimes^2 U_{g-1})$ of the Poincaré group $\mathscr{P}_0$ on bilinear symmetric maps A^2 from $E_\infty^\rho \to E_\infty^\rho$ and where $C^2 = R^{(+)2} - R^2$ is the cocycle defined by $R_g^2 = U_g^2(U^1_{g-1}\otimes U^1_{g-1})$ and $R_g^{(+)2} = U_g^{(+)2}(U^1_{g-1}\otimes U^1_{g-1})$, $g \in \mathscr{P}_0$. The modified wave operator equation shows that the cochain $R^{(+)2}$ has to be a cocycle and then that the cocycle C^2 has to be a coboundary. This is equivalent to the existence of a solution $U^{(+)}$, Ω_+ modulo terms of order at least three. There are similar equations for higher order terms: $\delta\Omega_+^n = C^n, \quad \delta R^{(+)n} = 0$, where C^n and $R^{(+)n}$ are functions of $\Omega_+^2,\ldots,\Omega_+^{n-1}$ and $U^{(+)2},\ldots,U^{(+)n}$.

There exist a modified wave operator and global solutions of the MD equations for a set of scattering data $(f,\dot{f},\alpha)$, satisfying $\hat{f}_\mu(k) = \hat{\dot{f}}_\mu(k) = 0$ for k in a neighbourhood of zero, i.e. f_μ and $\dot{f}_\mu$ have no low frequencies component. It follows that the usual wave operators do not exist, even in this case where there are no low frequencies.

Thus in the absence of low frequencies in the scattering data, for the electromagnetic potential, we can choose Ω_+ such that it intertwines the linear representation U^1 and the nonlinear representation U of $\mathscr{P}_0$. Now if $(f,\dot{f}) \in M_\infty^{\circ\rho}$, $1/2 < \rho < 1$, have nontrivial low frequency part (as for the Coulomb potential $(\hat{f}_\mu(k) \simeq |k|^{-2})$ and necessary to have asymptotic completeness) then there is a *cohomological obstruction* already for $n = 2$ if we want to obtain $U^{(+)2} = 0$. The cocycle R^2 can be split into a trivializable part $-C^2$ (a coboundary) and a nontrivializable part $R^{(+)2}$, which defines $U^{(+)2}$ and therefore the whole representation $U^{(+)}$. We call this nontrivial part the *infrared cocycle*. It is a coboundary, in particular, when $\hat{f}_\mu, \hat{\dot{f}}_\mu$ are equal to zero in a neighborhood of zero.

Physical Remarks. The asymptotic condition in Theorem 4.2 is (when $t \to \infty$) equivalent to

$$\|(A(t),\dot{A}(t)) - (A_L(t),\dot{A}_L(t))\|_{M^\rho} + \|\psi(t) - (e^{-i\varphi}\psi_L)(t)\|_D \to 0,$$

where $(A(t),\dot{A}(t),\psi(t)) = U_{\exp(tP_0)}(\Omega_+(u))$, $(A_L(t),\dot{A}_L(t),\psi_L(t)) = U^1_{\exp(tP_0)}u$ and for a fixed $u = (f,\dot{f},\alpha) \in \mathscr{O}^{(+)}_\infty$, φ is some function in $C^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$. Therefore $(A, e^{-i\varphi}\psi)$ converges to a free field (A_L,ψ_L). The *asymptotic representation $U^{(+)}$ is not linear*: it is equal to U^1 modulo a gauge projective map $Q_g = U^{(+)}_g U^1_{g^{-1}}$; i.e. this nonlinear transformation of the scattering data corresponds to a (Lorentz) gauge transformation $\varphi \to \varphi + \lambda_g$ of the asymptotic condition. But both the current density vector $j^\mu = \overline{\psi}\gamma^\mu\psi$ and the energy-momentum tensor for the MD system [given by $t^{\mu\nu} = -F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}(\overline{\psi}\gamma^\mu(i\partial^\nu - A^\nu)\psi + (\overline{(i\partial^\nu - A^\nu)\psi})\gamma^\mu\psi)$, where $0 \le \mu \le 3$, $0 \le \nu \le 3$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$] are invariant under gauge transformations $\psi' = e^{i\lambda}\psi$, $A'_\mu = A_\mu - \partial_\mu\lambda$ (not necessarily respecting the Lorentz gauge condition that gives here $\Box\lambda = 0$). Therefore they go to their free limits when $t \to \infty$. A similar discussion can be made for the relativistic angular momentum tensor.

The same methods can be used for nonabelian gauge theories (of the Yang-Mills type) coupled with fermions. The aim here is to separate asymptotically the linear (modulo an infrared problem that can be a lot worse in the nonabelian case) equation for the spinors from the pure Yang-Mills equation (the A_μ part). The next step would then be to linearize analytically the pure Yang-Mills equation (that is known to be formally linearizable), and then to combine all this with the deformation-quantization approach to deal rigorously with the corresponding quantum field theories.

The results presented here give indications how a true quantum field theory (i.e. not based on perturbative theory) can be developed on the basis of this first quantized (classical) field theory, dealing in particular with the infrared problem and the definition of observables. The quantization should be based on the mathematical facts found here and not on a nonrigourous perturbation theory developed from the free field by canonical quantization or using some algebraic postulates which (however interesting they may seem) reflect sometimes a "wishful thinking". In other words the path to follow should be based on "quantum deformations" (in the sense of star products) of the "classical" theory presented here. In this context it is important to get existence theorems for large initial data and to be able to localise specific solutions corresponding to large initial data, such as of the soliton or instanton type. In 4-dimensional space-time these are very hard problems, which should be no surprise: Problems worthy of attack prove their worth by hitting back!

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