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TITLE

Nonlinear Relativistic Wave Equations in General Dimensions

ABSTRACT

Given a linear partial differential evolution equation covariant under the action of a Lie group G, one can equate the evolution operator $\frac{\partial}{\partial t}$ with a representative of an element of the Lie algebra $\mathfrak{g}$ of G.

The same holds for a *nonlinear* P.D.E. covariant under G. Here the mathematical apparatus is a Lie theory of nonlinear representations. The linearizability of such equations is tightly connected to certain cohomology spaces.

After explaining these facts we shall concentrate on some scalar field equations and on a very recent result concerning *global* Cauchy problem, asymptotic completeness and infrared electron tail for the coupled Maxwell-Dirac (= electrodynamics) equations in physical $(3 + 1)$ dimensions.

Part I. NonLinear Representations

A. NonLinear Representations of Lie Groups and Algebras

Let E be a Banach or Fréchet space, $\mathcal{L}_n(E)$ the space of *continuous* **symmetric** *multilinear* maps $E^n \to E$. It is included in the space $\mathcal{L}(\hat{\otimes}_\pi^n E, E)$ of linear maps from a completed tensor product of n copies of E into E. To any $f^n \in \mathcal{L}_n(E)$ one associates the monomial $\hat{f}^n$: $E \to E$ by:

$$f^n(\varphi, \ldots, \varphi) \equiv \hat{f}^n(\varphi), \quad \varphi \in E.$$

We denote by $\mathcal{F}(E)$ the space of *formal series* $f = \sum_{n=1}^{\infty} f^n$ with product $\bullet$ given by the composition of maps from E to E, i.e. if $h = \sum_1^\infty h^n \in \mathcal{F}(E)$ then the n-th term of the product is given by:

$$E \ni \widehat{f \bullet h}^n(\varphi) = \sum_{1 \leq p \leq n} f^p\big(\sum_{i_1 + \ldots + i_p = n} \hat{h}^{i_1}(\varphi) \otimes \ldots \otimes \hat{h}^{i_p}(\varphi)\big)$$

A (formal)*nonlinear representation* of a Lie group G is (S, E) where S is a homomorphism of G into the group of *invertible* elements of $\mathcal{F}(E)$:

$$G \ni g \mapsto S_g = \sum_{n=1}^{\infty} S_g^n$$

and similarly one defines the notion of analytic nonlinear representation.

Equivalence of nonlinear representations:

$$(S, E) \sim (S', E' = E) \Leftrightarrow S'_g = A S_g A^{-1}, \quad A \in \mathcal{F}(E) \quad \text{invertible}$$

S is said *linearizable* if $S \sim S^1$, S^1 linear.

Associated linear maps: Take $\tilde{E} = \cup_n \oplus_{p=1}^{n} (\hat{\otimes}^p E)$ ("Fock space without vacuum"). A one-to-one algebra homomorphism $\Lambda : \mathcal{F}(E) \to \mathcal{L}(\tilde{E})$ mapping $\bullet$ to the product of linear maps is defined on the n-th level by:

$$\Lambda(f)^n = \sum_{1 \leq p \leq n} \sum_{i_1 + \ldots + i_p = n} (f^{i_1} \otimes \ldots \otimes f^{i_p}).$$

Its differential $d\Lambda$ is then defined on the n-th level by:

$$d\Lambda(f)^n = \sum_{1 \leq p \leq n} (\sum_{0 \leq q \leq p-1} I_q \otimes f^{n-p+1} \otimes I_{p-q-1})$$

and maps the bracket $f \times h - h \times f$ of formal series (which is therefore a Lie bracket) into the commutator of linear operators; $f \times h$ is defined on the n-th level by:

$$(f \times h)^n = \sum_{1 \leq p \leq n} f^p \circ (\sum_{0 \leq q \leq p-1} I_q \otimes h^{n-p+1} \otimes I_{p-q-1}).$$

A *nonlinear representation of a Lie algebra* $\mathfrak{g}$ is then a Lie morphism of $\mathfrak{g}$ into $\mathcal{F}(E)$ equipped with this bracket.

An important result of the theory is that one has a *Lie theory*, i.e. that on the *differentiable vectors of the linear part* S^1,

$$E^\infty = \text{span of} \quad \{\varphi = \int_G S^1_g \psi \alpha(g) dg\}, \quad \psi \in E, \quad \alpha \in \mathcal{D}(G)$$

$$S \quad \text{NL rep. of} \quad G \quad \Longleftrightarrow \quad dS \quad \text{NL rep. of} \quad \mathfrak{g}.$$

1. *Cohomology.*
G a group, U a linear rep. of G in E, $g, g' \in G$:
1-cocycle: map $R : G \to E$, $R(gg') = R(g) + U(g)R(g') \Rightarrow \quad Z^1(G, E)$.
1-coboundary: $\exists$ (0-cochain)$\varphi \in E$, $R(g) = U(g)\varphi - \varphi \quad \Rightarrow \quad B^1(G, E)$.
1-cohomology: $H^1(G, E) = Z^1(G, E)/B^1(G, E)$ ("affine").

2. *Extension* (V, F) of (U, E) by (U', E'):
exact sequence of G-modules $\quad 0 \to E \to F \to E' \to 0$.
An extension is given by a $1-$ *cocycle* of G for the representation T in $\mathcal{L}(E', E)$ defined by:

$$\mathcal{L}(E', E) \ni A \mapsto T(g)A = U(g)AU'(g^{-1}) \in \mathcal{L}(E', E).$$

3. *NL reps. as successive extensions.*
Let (S, E) be a NL rep. of G, $S = \sum_{n=1}^{\infty} S^n$. Then:

$$S^2(g, g') = S^1(g) \circ S^2(g') + S^2(g) \circ (S^1(g') \otimes S^1(g').$$

Thus $R(g) \equiv S^1(g) \circ S^2(g^{-1})$ is a 1-cocycle of G in $\mathcal{L}(\otimes^2 E, E)$ for the representation T:

$$\mathcal{L}(\otimes^2 E, E) \ni A \mapsto T(g)A = S^1(g) \circ A \circ (S^1(g^{-1}) \otimes S^1(g^{-1}).$$

Thus (by Λ) on $\tilde{E}$, S is obtained by *successive extensions* of S^1 by its (symmetric) tensorial powers $\otimes^n S^1$, $n \geq 2$: first S^1 by $S^1 \otimes S^1$, then the result by $\otimes^3 S^1$ etc. Thus one has:
4. Theorem. *If (S, E) is a NL rep. of a Lie group G in a Fréchet space of differentiable vectors, and if the (differentiable) cohomology spaces $H^1(G, \mathcal{L}(\otimes^n E, E)) = 0 \quad \forall n \geq 2, \quad S$ is linearizable.*
Scheme of proof: $R(g^{-1}) = S^1(g^{-1}) \circ S^2(g)$ is a 1-cocycle in $Z^1(G, \mathcal{L}(\otimes^2 E, E))$, therefore it is the coboundary of some

$$B^2 \in \mathcal{L}(\otimes^2 E, E) : \quad S^2(g) = S^1(g) \circ B^2 - B^2 \circ (\otimes^2 S^1(g))$$

and S is equivalent to a representation

$$g \mapsto (I - B^2)^{-1} S(g)(I - B^2) = S^1(g) + [\text{terms of order} \geq 3]$$

and so on, so that we get the linearizing operator $\prod_{k=2}^{\infty}(I - B^k)$.

Part II. NonLinear Klein-Gordon equation

A. The Equation

$$(\Box + m^2)\varphi(t,x) = P(\varphi(t,x), \frac{\partial}{\partial t}\varphi(t,x)\nabla\varphi(t,x))$$

where $m^2 > 0$, $x \in \mathbb{R}^n$, $n \geq 2$ and P is analytic with no constant and no linear term ($P(0) = 0 = dP(0)$).

We transform it into an **Evolution Equation** by setting $\varphi(t)(x) = \varphi(t,x)$; $(\dot\varphi(t))(x) = \frac{\partial}{\partial t}\varphi(t,x)$; and defining $a_\varepsilon(t) = \dot\varphi(t) + \varepsilon i\delta\varphi(t)$, where $\varepsilon = \pm 1$ and $\delta = (m^2 - \Delta)^{1/2}$ and $F(a(t)) = P(\varphi(t), \dot\varphi(t), \nabla\varphi(t))$:

$$\frac{d}{dt}\begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix} = i\delta\begin{pmatrix} a_+(t) \\ -a_-(t) \end{pmatrix} + \begin{pmatrix} F(a(t)) \\ F(a(t)) \end{pmatrix}$$

In the following we write $\binom{a_+}{a_-}$ as (f_+, f_-).

B. Associated NonLinear Representation of the Poincaré Lie Algebra

The space is $E = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$, with differentiable vectors space $E^\infty = \mathcal{S}(\mathbb{R}^n) \oplus \mathcal{S}(\mathbb{R}^n)$. We denote by Π the usual basis of the Lie algebra $\mathfrak{p}$, and by Π' that of the enveloping algebra $\mathcal{U}(\mathfrak{p})$ defined by the Poincaré-Birkhoff-Witt theorem. The **linear representation** of $\mathfrak{p}$ is defined by:

$T^1_{P_0}(f_+, f_-) = i\delta(f_+, -f_-)$; $\quad T^1_{P_j} = \partial_j \ (1 \leq j \leq n)$

$T^1_{M_{ij}} = (x_i\partial_j - x_j\partial_i)$, $\quad T^1_{M_{0j}}(f_+, f_-) = (i\delta x_j f_+, -i\delta x_j f_-)$

and it exponentiates to a linear representation U^1 of the Poincaré group $\mathcal{P}$ on E^∞ and on E.

The **nonlinear representation** of $\mathfrak{p}$ is then defined by:

$T_X = T^1_X + \tilde{T}_X, \quad X \in \mathfrak{p} \quad$ with $T^1_X \in \mathcal{L}(E^\infty)$ and $T_X : E^\infty \to E^\infty$

$$\tilde{T}_{P_0}(f) = (F(f), F(f))$$

$$\tilde{T}_{P_i}(f) = 0 = \tilde{T}_{M_{ij}}(f), \quad \tilde{T}_{M_{0j}}(f) = (x_j F(f), x_j F(f))$$

$$\text{NLKG}: \quad \frac{d}{dt}f(t) = (T^1_{P_0} + \tilde{T}_{P_0})f(t)$$

C. The Spaces

$E_i =$ completion of E^∞ for the norm

$$\|f\|_{E_i} = (\Sigma_{\substack{Y\in\Pi' \\ |Y|\le i}} \|T^1_Y(f)\|^2_E)^{1/2}$$

Local solutions are obtained by classical methods: let $\mathcal{V}$ be a neighborhood of the identity in $\mathcal{P}$; if $n \ge 2$ and $N \ge N_0$, $\exists \mathcal{O}^0_{N_0}$ neighborhood of 0 in E_{N_0} and a local analytic action T of $\mathcal{P}$: $\mathcal{V}\times(E_N \cap \mathcal{O}^0_{N_0}) \to E_N$.
Global solutions follow from the linearizability of time translations.

D. Yang-Feldman-Källén (YFK) Equation.

Write $U_t = T_{\exp tP_0}$, $\quad U^1_t = \exp(tT^1_{P_0})$ (Poincaré time translations).
Theorem. *Linearisability of U_t plus asymptotic freedom $\iff$ YFK*:
If $\exists$ operator A (invertible in 0) on a neighborhood $\mathcal{O}^0_{N_0}$ of 0 in E_{N_0} with $U_tA = AU^1_t$ and $\lim\limits_{t\to\infty}\|U_tAf - U^1_t f\|_E = 0$, then this A is solution of the YFK equation:

$$A = I - \int_0^\infty U^1_{-s}\tilde{T}_{P_0}AU^1_s ds$$

and conversely (if $\exists A$ solution of YFK, one has linearizability and asymptotic freedom).
Proof (heuristic) $\Longrightarrow$: $\quad h(t) = U^1_{-t}U_tAf - f \to 0 \quad$ for $t\to\infty$

$$\frac{d}{dt}h(t) = U^1_{-t}\tilde{T}_{P_0}U_tAf \Rightarrow h(t) = \int_0^t U^1_{-s}\tilde{T}_{P_0}AU^1_s f ds + Af - f$$

and make $t\to\infty$: YFK.

$$\Longleftarrow: \quad \text{YFK gives} \quad AU^1_t - U^1_t A = U^1_t \int_0^t U^1_{-s}\tilde{T}_{P_0}AU^1_s ds$$

Therefore, if $g(t) = AU^1_t f$, one has $\frac{d}{dt}g(t) = (T^1_{P_0} + \tilde{T}_{P_0})g(t)$ and $g(0) = Af$. But U_tAf is solution of the same equation with the same initial conditions, thus $U_tA = AU^1_t$. Also

$$AU^1_t f - U^1_t f = \int_t^\infty U^1_t U^1_{-s}\tilde{T}_{P_0}AU^1_s f ds \to 0$$

$$(\text{if } \textstyle\int_0^\infty \|U^1_{t-s}\tilde{T}_{P_0}AU^1_s f\|ds < \infty)$$

E. Application.

Write $A = \sum_{k\geq 1} A^k$, $A^k \in \mathcal{L}^k(E_{N_0})$ (symmetric continuous multilinear maps).

$$\begin{aligned} \text{with}\quad & A^1 = \mathrm{Id}_{E_{N_0}}. \\ \text{Then}\quad & A^2 = -\int_0^\infty U^1_{-s} T^2_{P_0} U^1_s \, ds \\ & A^3 = -\int_0^\infty U^1_{-s}\big(T^3_{P_0} + T^2_{P_0}(I \otimes A^2 + A^2 \otimes I)\big) U^1_s \, ds \\ & \text{etc.} \end{aligned}$$

Problem: prove convergence (existence of solution of YFK).

F. Convergence

The problem is to control $||A^k U^1_t||$ for $t \to \infty$. One has (with $\partial_i = \partial/\partial x_i$):

$$||U^1_t f||_{L^\infty} \leq |t|^{-\frac{n}{2}} c(f) \quad \text{as} \quad t \to \infty$$
$$\text{with} \quad c(f) = c(||f||_{L^1} + \sum_i ||\partial_i f||_{L^1} + \sum_{i,j} ||\partial_i \partial_j f||_{L^1})$$

Therefore, if $n = 4$, one can compute A.

For $n = 3$ one needs $P''(0) = 0$ (no quadratic term).

For $n = 2$ one needs $P''(0) = 0 = P'''(0)$ ($n = 1$ is atypical).

But *using the covariance of NLKG one gets convergence for* $n \geq 2$ (with any P, $P(0) = 0 = dP(0)$). Indeed, if $X, Y \in \mathfrak{p}$,

$$T^2_{[X,Y]} = T^1_X T^2_Y - T^2_Y(T^1_X \otimes I + I \otimes T^1_X) - (T^1_Y T^2_X - T^2_X(T^1_Y \otimes I + I \otimes T^1_Y)),$$

i.e. $T^2 \in Z^1(\mathfrak{p}, \mathcal{L}^2(E^\infty))$ (cocycle).

T^2 extends to $\mathcal{U}(\mathfrak{p})$ with

$$T^2_{xy} = T^1_x T^2_y + T^2_x d(U^1 \otimes U^1)_y. \tag{1}$$

For the Casimir $y = P_0^2 - \sum\limits_{i=1} P_i^2 = P_\mu P^\mu$: $\qquad xy = yx \ \ \forall x.$

(1) then gives $T^2_x = T^1_x(T^2_y R^{-1}) - (T^2_y R^{-1}) d(U^1 \otimes U^1)_x$

because $T^1_y = T^1_{P_\mu P^\mu} = m^2$ and $R^{-1} = (m^2 - d(U^1 \otimes U^1)_x)^{-1}$ is defined

(since $|m^2 - (p_1^0 + p_2^0)^2 + \sum\limits_{i=1}^{n} (p_1^i + p_2^i)^2| \geq m^2$ for $n = 2, 3 \ldots$).

Thus if we take $A^2 = T^2_y R^{-1}$ one has until second order:

$$T_g(I + A^2) = (I + A^2) T^1_g$$

$$\begin{aligned}
&||A^2(U_t g \otimes U_t f)||_E \leq C_{g,f}(1 + |t|)^{-n/2} \\
&||A^2(U_t g \otimes U_t f)||_W \leq C_{g,f}(1 + |t|)^{-n} \\
&\qquad \text{for } t \in \mathbb{R}, \quad g, f \in W = W^{k,\infty}(\mathbb{R}^n) \oplus W^{k,\infty}(\mathbb{R}^n)
\end{aligned}$$

etc. $\Rightarrow \exists A$ solution of YFK.

G. Theorem (existence and uniqueness of global solutions)

$\exists N_0 \geq 0$ and neighborhood $\mathcal{O}^\alpha_{N_0}$ $(\alpha = -1, 0, 1)$ of 0 in E_{N_0} such that (write $\mathcal{O}^\alpha_N = E_N \cap \mathcal{O}^\alpha_{N_0}$ for $N \geq N_0$, $\quad \mathcal{O}^\alpha_\infty = E^\infty \cap \mathcal{O}^\alpha_{N_0}$):

1) T is analytic map $\mathcal{P}_\times \mathcal{O}^0_{N_0} \to \mathcal{O}^0_{N_0}$ and $g \mapsto T^1_{g^{-1}} T_g$ is continuous $\mathcal{P} \to \mathcal{H}(\mathcal{O}^0_{N_0}, E_{N_0})$ (space of Banach analytic maps).
2) $\exists$ a unique analytic $A_\varepsilon : \mathcal{O}^\varepsilon_{N_0} \to \mathcal{O}^0_{N_0}$ $(\varepsilon = \pm 1)$ mapping $\mathcal{O}^\varepsilon_N$ in $\mathcal{O}^0_N$ for $N \geq N_0$ such that $T_g A_\varepsilon = A_\varepsilon T^1_g$ for $g \in \mathcal{P}$ (Poincaré group) and
3) $\lim\limits_{t \to \varepsilon\infty} \parallel T_{\exp tP_0} A_\varepsilon \theta - T^1_{\exp tP_0} \theta \parallel_\varepsilon = 0, \quad \theta \in \mathcal{O}^\varepsilon_{N_0}.$

Therefore if $(\varphi_0, \dot{\varphi}_0) \in \mathcal{O}^-_\infty \cap \mathcal{O}^+_\infty$ there exists a unique solution $(\varphi_t, \dot{\varphi}_t)$ of NLKG in $\mathcal{S}(\mathbb{R}^n) \oplus \mathcal{S}(\mathbb{R}^n)$ for all $t \in \mathbb{R}$.

Part III. Asymptotic completeness, global existence and the infrared problem for the Maxwell-Dirac equations

Mathematical Framework

1. The Equations. We use conventional notations: electron charge $e = 1$; Dirac matrices γ^μ, $0 \leq \mu \leq 3$; metric tensor $g^{\mu\nu}$, $g^{00} = 1$, $g^{ii} = -1$ for $1 \leq i \leq 3$ and $g^{\mu\nu} = 0$ for $\mu \neq \nu$; $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$; $\partial_\mu = \partial/(\partial y^\mu)$; $\Box = \partial_\mu\partial^\mu$. The classical Maxwell-Dirac (M-D) equations read:

$$\Box A_\mu = \overline{\psi}\gamma_\mu\psi, \quad 0 \leq \mu \leq 3, \tag{1.1a}$$

$$(i\gamma^\mu\partial_\mu + m)\psi = A_\mu\gamma^\mu\psi, \quad m > 0, \tag{1.1b}$$

$$\partial_\mu A^\mu = 0, \tag{1.1c}$$

where $\overline{\psi} = \psi^+\gamma_0$, ψ^+ being the Hermitian conjugate of the Dirac spinor ψ.

We write equations (1.1a) and (1.1b) as an evolution equation:

$$\frac{d}{dt}(A_\mu(t), \dot{A}_\mu(t)) = (\dot{A}_\mu(t), \Delta A_\mu(t)) + (0, \overline{\psi}(t)\gamma_\mu\psi(t)), \tag{1.2a}$$

$$\frac{d}{dt}\psi(t) = \mathcal{D}\psi(t) - iA_\mu(t)\gamma^0\gamma^\mu\psi(t), \tag{1.2b}$$

where $\mathcal{D} = -\sum_{j=1}^{3}\gamma^0\gamma^j\partial_j + i\gamma^0 m$, $\Delta = \sum_{j=1}^{3}\partial_j^2$, $t \in \mathbb{R}$ and where $A_\mu(t)$, $\dot{A}_\mu(t)\colon \mathbb{R}^3 \to \mathbb{R}$, $\psi(t)\colon \mathbb{R}^3 \to \mathbb{C}^4$. The Lorentz gauge condition (1.1c) takes on initial conditions $A_\mu(t_0)$, $\dot{A}_\mu(t_0)$, $0 \leq \mu \leq 3$, and $\psi(t_0)$ at $t = t_0$ the form (with $A^\mu = g^{\mu\nu}A_\nu$):

$$\dot{A}^0(t_0) + \sum_{i=1}^{3}\partial_i A^i(t_0) = 0, \quad \Delta A^0(t_0) + |\psi(t_0)|^2 + \sum_{i=1}^{3}\partial_i\dot{A}^i(t_0) = 0, \tag{1.3}$$

.

2. Poincaré covariance: linear part and representation spaces. Equations (1.1a), (1.1b) and (1.1c) are manifestly covariant under the action of the universal covering $\mathcal{P}_0 = \mathbb{R}^4 \otimes SL(2,\mathbb{C})$ of the Poincaré group, therefore one can complete the time translation generator, formally defined by (1.2a) and (1.2b), to a nonlinear representation of the whole Lie algebra $\mathfrak{p} = \mathbb{R}^4 \oplus \mathfrak{sl}(2,\mathbb{C})$ of $\mathcal{P}_0$.

Topological vector spaces of the representations. M^ρ, $-1/2 < \rho < \infty$ is the completion of $S(\mathbb{R}^3,\mathbb{R}^4) \oplus S(\mathbb{R}^3,\mathbb{R}^4)$ with respect to the norm

$$\|(f,\dot f)\|_{M^\rho} = \big(\||\nabla|^\rho f\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)} + \||\nabla|^{\rho-1}\dot f\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)}\big)^{1/2}, \tag{1.4a}$$

where $|\nabla| = (-\Delta)^{1/2}$. Denote $D = L^2(\mathbb{R}^3,\mathbb{C}^4)$ and $E^\rho = M^\rho \oplus D$, the Hilbert space with norm

$$\|(f,\dot f,\alpha)\|_{E^\rho} = \big(\|(f,\dot f)\|^2_{M^\rho} + \|\alpha\|^2_D\big)^{1/2}. \tag{1.4b}$$

$M^{\circ\rho}$ is the closed subspace of $(f,\dot f) \in M^\rho$, $f = (f_0, f_i)$, such that

$$\dot f_0 = \sum_{1\le i\le 3} \partial_i f_i, \quad f_0 = -\sum_{1\le i\le 3} |\nabla|^{-2}\partial_i \dot f_i, \tag{1.4c}$$

A solution B_μ of $\square B_\mu = 0$, $0 \le \mu \le 3$, with initial conditions $(f,\dot f) \in M^\rho$ satisfies the gauge condition $\partial_\mu B^\mu = 0$ if and only if $(f,\dot f) \in M^{\circ\rho}$. We define $E^{\circ\rho} = M^{\circ\rho} \oplus D$.

Let $\Pi = \{P_\mu, M_{\alpha\beta} \big| 0 \le \mu \le 3, 0 \le \alpha < \beta \le 3\}$ be a standard basis of the Poincaré Lie algebra. There is a linear continuous representation U^1 of $\mathcal{P}_0$ in E^ρ, with differentiable vectors E^ρ_∞, leaving $E^{\circ\rho}$ invariant, the differential of which is the following linear representation T^1 of $\mathfrak{p}$ in E^ρ_∞:

$$T^1_{P_0}(f,\dot f,\alpha) = (\dot f, \Delta f, \mathcal{D}\alpha), \tag{1.5a}$$

$$T^1_{P_i}(f,\dot f,\alpha) = \partial_i(f,\dot f,\alpha), \quad 1 \le i \le 3, \tag{1.5b}$$

$$(T^1_{M_{ij}}(f,\dot f,\alpha))(x) = -(x_i\partial_j - x_j\partial_i)(f,\dot f,\alpha)(x) + (n_{ij}f, n_{ij}\dot f, \sigma_{ij}\alpha)(x) \tag{1.5c}$$

$$1 \le i < j \le 3, \sigma_{ij} = 1/2\gamma_i\gamma_j \in \mathfrak{su}(2), n_{ij} \in \mathfrak{so}(3),$$

$$(T^1_{M_{0i}}(f,\dot f,\alpha))(x) = \big(x_i\dot f(x), \sum_{j=0}^{3}\partial_j\big(x_i\partial_j f(x)\big), x_i\mathcal{D}\alpha(x)\big) + (n_{0i}f, n_{0i}\dot f, \sigma_{0i}\alpha)(x),$$

$$1 \le i \le 3, \quad \sigma_{0i} = 1/2\gamma_0\gamma_i, \quad n_{0i} \in \mathfrak{so}(3,1), \tag{1.5d}$$

We introduce the graded sequence of Hilbert spaces E^ρ_i, $i \geq 0$, $E^\rho_0 = E^\rho$, where $E^\rho_\infty \subset E^\rho_j \subset E^\rho_i$ for $i \leq j$. E^ρ_i is the space of C^i-vectors of the representation U^1.

Given an ordering $X_1 < X_2 < \cdots < X_{10}$ on Π, in the universal enveloping algebra $U(\mathfrak{p})$ of $\mathfrak{p}$, the subset of all products $X_1^{\alpha_1} \cdots X_{10}^{\alpha_{10}}$, $0 \leq \alpha_i$, $1 \leq i \leq 10$, is a basis Π' of $U(\mathfrak{p})$.

E^ρ_i, $i \in \mathbb{N}$ is the completion of E^ρ_∞ with respect to the norm

$$\|u\|_{E^\rho_i} = \Big(\sum_{\substack{Y \in \Pi' \\ |Y| \leq i}} \|T^1_Y(u)\|^2_{E^\rho} \Big)^{1/2}, \tag{1.6a}$$

$M^{\circ\rho}_i = M^\rho_i \cap M^{\circ\rho}$, $E^{\circ\rho}_i = E^\rho_i \cap E^{\circ\rho}$, $M^{\circ\rho}_\infty = M^\rho_\infty \cap M^{\circ\rho}$ and $E^{\circ\rho}_\infty = E^\rho_\infty \cap E^{\circ\rho}$, where M^ρ_i (resp. M^ρ_∞) is the image of E^ρ_i (resp. E^ρ_∞) in M^ρ under the canonical projection of E^ρ on M^ρ. Let D_i (resp. D_∞) be the image of E^ρ_i (resp. E^ρ_∞) in D under the canonical projection of E^ρ on D. $D_\infty = S(\mathbb{R}^3, \mathbb{C}^4)$. M^ρ_∞ contains long-range potentials.

3. Nonlinear Poincaré covariance. A nonlinear representation T of $\mathfrak{p}$ in E^ρ_∞ is obtained by the fact that the M-D equations are covariant:

$$T_X = T^1_X + T^2_X, \quad X \in \mathfrak{p}, \tag{1.7}$$

where T^1 is given by (1.5a)–(1.5d) and, for $u = (f, \dot{f}, \alpha) \in E^\rho_\infty$,

$$T^2_{P_0}(u) = (0, \overline{\alpha}\gamma\alpha, -if_\mu\gamma^0\gamma^\mu\alpha), \quad \gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3), \tag{1.8a}$$

$$T^2_{P_i} = 0 \quad \text{for } 1 \leq i \leq 3, \quad T^2_{M_{ij}} = 0 \quad (1 \leq i < j \leq 3), \tag{1.8b}$$

$$(T^2_{M_{0j}}(u))(x) = x_j (T^2_{P_0}(u))(x), \quad 1 \leq j \leq 3, \quad x = (x_1, x_2, x_3). \tag{1.8c}$$

The gauge condition (1.1c) takes on initial data $u = (f, \dot{f}, \alpha)$ the form

$$\dot{f}_0 = \sum_{1 \leq i \leq 3} \partial_i f_i, \qquad \Delta f_0 = \sum_{1 \leq i \leq 3} \partial_i \dot{f}_i - |\alpha|^2. \tag{1.1c$'$}$$

The subspace V^ρ_∞, $1/2 < \rho < 1$ of elements in E^ρ_∞ which satisfy these gauge conditions is diffeomorphic to $E^{\circ\rho}_\infty$.

The problem to integrate globally the nonlinear Lie algebra representation T therefore consists of proving the existence of an open neighbourhood $\mathcal{U}_\infty$ of zero in V^ρ_∞ and a group action $U\colon \mathcal{P}_0 \times \mathcal{U}_\infty \to \mathcal{U}_\infty$, which is C^∞ and such that $U_g(0) = 0$ for $g \in \mathcal{P}_0$ and $\frac{d}{dt}\, U_{\exp(tX)}(u) \mid_{t=0} = T_X\, u, X \in \mathfrak{p}$.

We extend T from $\mathfrak{p}$ to to the enveloping algebra $U(\mathfrak{p})$ by defining inductively: $T_{\mathbb{I}} = I$, where $\mathbb{I}$ is the identity element in $U(\mathfrak{p})$, and

$$T_{YX} = DT_Y.T_X, \quad Y \in U(\mathfrak{p}), X \in \mathfrak{p}. \tag{1.9}$$

where $(DA.B)(f)$ is the Fréchet derivative of A at the point f in the direction $B(f)$. It maps E^ρ_∞ to E^ρ_∞. Moreover

$$\frac{d}{dt} T_{Y(t)}(u(t)) = T_{P_0 Y(t)}(u(t)), \quad Y \in U(\mathfrak{p},) \tag{1.10}$$

where

$$Y(t) = \exp(t\,\mathrm{ad}_{P_0})Y, \quad \mathrm{ad}_{P_0} Z = [P_0, Z], \tag{1.11}$$

if

$$\frac{d}{dt} u(t) = T_{P_0}(u(t)). \tag{1.12}$$

We also define $\tilde{T}_Y$, $Y \in U(\mathfrak{p})$ by $T_Y = T^1_Y + \tilde{T}_Y$ and denote by T^{M1} or T^{1M} (resp. T^{D1} or T^{1D}) the linear representation of $\mathfrak{p}$ which is the restriction of the linear representation T^1 to M^ρ (resp. D), etc.

The Infrared Problem

On the classical level the infrared problem consists of determining to which extent the long-range interaction created by the coupling $A^\mu j_\mu$ between the electromagnetic potential A_μ and the current $j_\mu = \overline{\psi}\gamma_\mu\psi$ is an obstruction for the separation, when $|t| \to \infty$, of the nonlinear relativistic system into two asymptotic isolated relativistic systems, one for the electromagnetic potential A_μ and one for the Dirac field ψ. It will be proved here that there is such an obstruction, which in particular implies that *asymptotic in and out states do not transform according to a linear representation of the Poincaré group.* This constitutes a serious problem for the second quantization of the asymptotic (in and out going) fields.

Indeed the particle interpretation usually requires free relativistic fields, i.e. at least a linear representation of the Poincaré group (U^1 in our case). Therefore we have to introduce nonlinear representations $U^{(-)}$ and $U^{(+)}$ of $\mathcal{P}_0$, which can give the transformation of the asymptotic in and out states under $\mathcal{P}_0$ and permit a particle interpretation.

Two reasons permit to determine the class of asymptotic representations $U^{(\varepsilon)}$, $\varepsilon = \pm$. First, the classical observables, 4-current density, 4-momentum and 4-angular momemtum are invariant under gauge transformations. Second, if the evolution equations become linear after a gauge transformation one can use the freedom of gauge in the second quantization of the fields. It is therefore reasonable to postulate that the asymptotic representations $g \mapsto U_g^{(\varepsilon)}$ are linear modulo a nonlinear gauge transformation depending on g and respecting the Lorentz gauge condition.

Moreover to determine the class of admissible modified wave operators it is reasonable to postulate that solutions of the M-D equations (1.1a)–(1.1c) should converge when $t \to \pm\infty$ in E^ρ to free solutions (i.e. solutions of equations (1.1a)–(1.1c) but with vanishing right-hand side) modulo a gauge transformation, not necessarily respecting the Lorentz gauge condition; such transformations are admissible since they leave invariant the observables. In mathematical terms the infrared problem of the M-D equations then consists of determining diffeomorphisms $\Omega_\varepsilon \colon \mathcal{O}_\infty^{(\varepsilon)} \to \mathcal{U}_\infty$, $\varepsilon = \pm$, the **modified wave operators**, where $\mathcal{O}_\infty^{(\varepsilon)}$ is an open neighbourhood of zero in $E_\infty^{\circ\rho}$ and where $\mathcal{U}_\infty$ is a neighbourhood of zero in V_∞^ρ, satisfying

$$U_g^{(\varepsilon)} = \Omega_\varepsilon^{-1} \circ U_g \circ \Omega_\varepsilon, \quad g \in \mathcal{P}_0, \varepsilon = \pm, \tag{1.14}$$

where the asymptotic representations are C^∞ functions $U^{(\varepsilon)} \colon \mathcal{P}_0 \times E_\infty^{\circ\rho} \to E_\infty^{\circ\rho}$. In order to satisfy the two preceding postulates, we impose supplementary conditions on $U^{(\varepsilon)}$ and Ω_ε, which we shall justify at the end. Denote the Fourier transformation $f \mapsto \hat{f}$. The orthogonal projections $P_\varepsilon(-i\partial)$ in D on initial data with energy sign ε, $\varepsilon = \pm$, for the Dirac equation are:

$$(P_\varepsilon(-i\partial)\alpha)^\wedge(k) = P_\varepsilon(k)\hat{\alpha}(k) = \frac{1}{2}\Big(I + \varepsilon\Big(-\sum_{j=1}^{3} \gamma^0\gamma^j k_j + m\gamma^0\Big)\omega(k)^{-1}\Big)\hat{\alpha}(k), \tag{1.16}$$

where $\omega(k) = (m^2 + |k|^2)^{1/2}$.

We postulate that the asymptotic representations have the following form:

$$U_g^{(+)}(u) = (U_g^{(+)M}(u), U_g^{(+)D}(u)), \quad U^{(+)M} = U^{1M}, \tag{1.17a}$$

$$(U_g^{(+)D}(u))^\wedge(k) = \sum_{\varepsilon=\pm} e^{i\varphi_g(u,-\varepsilon k)} P_\varepsilon(k)(U_g^{1D}\alpha)^\wedge(k), \ \ g \in \mathcal{P}_0, \tag{1.17b}$$

where $u = (f, \dot{f}, \alpha) \in E_\infty^{\circ\rho}$, the function $(g, u, k) \mapsto \varphi_g(u, -\varepsilon k)$ from $\mathcal{P}_0 \times E_\infty^{\circ\rho} \times \mathbb{R}^3$ to $\mathbb{R}$ is C^∞ and if $(h_g(u))(t,x) = \varphi_g(u, mx(t^2 - |x|^2)^{-1/2})$, $t > 0$, $|x| < t$, then $\Box h_g(u) = 0$. Finally we impose the asymptotic condition

$$\begin{aligned} &\|U^M_{\exp(tP_0)}(\Omega_+(u)) - U^{1M}_{\exp(tP_0)}(f,\dot{f})\|_{M^\rho} \\ &\quad + \|U^D_{\exp(tP_0)}(\Omega_+(u)) - \sum_{\varepsilon=\pm} e^{is_\varepsilon^{(+)}(u,t,-i\partial)} P_\varepsilon(-i\partial) U^{1D}_{\exp(tP_0)}\alpha\|_D \to 0, \end{aligned} \tag{1.17c}$$

when $t \to \infty$, where $u = (f, \dot{f}, \alpha) \in \mathcal{U}_\infty$, $(u,t,k) \mapsto s_\varepsilon^{(+)}(u,t,k)$ is a C^∞ function from $\mathcal{U}_\infty \times \mathbb{R} \times \mathbb{R}^3$ to $\mathbb{R}$, $s_\varepsilon^{(+)}(u,t,-i\partial)$ is the operator defined by inverse Fourier transform of the multiplication operator $k \mapsto s_\varepsilon^{(+)}(u,t,k)$. It is possible to choose

$$s_\varepsilon^{(+)}(u,t,k) = -\vartheta\big(A^{(+)}(u), (t, -\varepsilon tk/\omega(k))\big), \tag{1.18}$$

where $A^{(+)}(u)$ is a certain approximate solution of the M-D equations absorbing the long-range part of A for a solution (A,ψ), and where $\vartheta(H,y) = \int_{L(y)} H_\mu(z) dz^\mu$, $\quad y \in \mathbb{R}^4, H\colon \mathbb{R}^4 \to \mathbb{R}^4$ and $L(y) = \{z \in \mathbb{R}^4 \big| z = sy, 0 \leq s \leq 1\}$. The function $(t,k) \mapsto s_\varepsilon^{(+)}(u,t,k)$ was determined by the fact that $S_\varepsilon(u,t,k) = \varepsilon\omega(k)t + s_\varepsilon^{(+)}(u,t,k)$ has to be in a certain sense an approximate solution of the *Hamilton-Jacobi equation* for a relativistic electron in an external electromagnetic potential:

$$\begin{aligned} &\big(\frac{\partial}{\partial t}\mathcal{S}_\varepsilon(u,t,k) + A_0(t, -\nabla_k \mathcal{S}_\varepsilon(u,t,k))\big)^2 \\ &\quad - \sum_{i=1}^{3} \big(k_i + A_i(t, -\nabla_k \mathcal{S}_\varepsilon(u,t,k))\big)^2 = m^2. \end{aligned} \tag{1.20a}$$

Some more notations. $u_+ = (f, \dot{f}, \alpha) \in E^{\circ\rho}_\infty$ and

$$J^{(+)}_\mu(t,x) = (m/t)^3 \big(\omega(q(t,x))/m\big)^5 \sum_{\varepsilon=\pm} ((P_\varepsilon(-i\partial)\alpha)^\wedge(\varepsilon q(t,x)/m))^+ \gamma^0\gamma_\mu((P_\varepsilon(-i\partial))\alpha)^\wedge(\varepsilon q(t,x)/m), \tag{1.21}$$

for $t > 0$, $|x| < t$, where $q(t,x) = -mx/(t^2 - |x|^2)^{1/2}$ and, for $|x| > 0$, $0 \leq t \leq |x|$, $J^{(+)}_\mu(t,x) = 0$. We choose a suitable cut-off function χ and introduce $A^{(+)} = A^{(+)1} + A^{(+)2}$ by

$$\left.\begin{aligned} (A^{(+)1}(t))(x) &= \chi_0((t^2 - |x|^2)^{1/2})(B^{(+)1}(t))(x) \\ (A^{(+)2}(t))(x) &= \chi((t^2 - |x|^2)^{1/2})(B^{(+)2}(t))(x), \end{aligned}\right\} \quad \text{for } t \geq |x|,$$

and (1.22a)

$$\left.\begin{aligned} (A^{(+)1}(t))(x) &= \chi_0(0)(B^{(+)1}(t))(x), \\ (A^{(+)2}(t))(x) &= 0, \end{aligned}\right\} \quad \text{for } t < |x|,$$

where

$$B^{(+)1}_\mu(t) = \cos(|\nabla|t) f_\mu + |\nabla|^{-1}\sin(|\nabla|t)\dot{f}_\mu, \quad t \in \mathbb{R}, \tag{1.22b}$$

$$B^{(+)2}_\mu(t) = -\int_t^\infty |\nabla|^{-1}\sin(|\nabla|(t-s))J^{(+)}_\mu(s,\cdot)ds, \quad t > 0. \tag{1.22c}$$

$U^{(+)}$ is defined by formulas (1.17a) and (1.17b) and by $\varphi_g = 0$ for g in $SL(2,\mathbb{C})$ and

$$\varphi_g(u, -\varepsilon k) = \vartheta^\infty\big((\chi_0 \circ \rho)(B^{(+)1}(\cdot + a) - B^{(+)1}(\cdot)), (\omega(k), -\varepsilon k)\big), \tag{1.23a}$$

for $g = \exp(a^\mu P_\mu)$, $k \in \mathbb{R}^3$, where

$$\vartheta^\infty(H, y) = \int_0^\infty y^\mu H_\mu(sy)ds, \quad y \in \mathbb{R}^4, \text{ and } \rho(t,x) = (t^2 - |x|)^{1/2}. \tag{1.23b}$$

Statements.

Theorem I. *Let $1/2 < \rho < 1$. If $n \geq 4$ then $U^{(+)}\colon \mathcal{P}_0 \times E_n^{\circ\rho} \to E_n^{\circ\rho}$ is a continuous nonlinear representation of $\mathcal{P}_0$ in $E_n^{\circ\rho}$ and, in addition, the function $U^{(+)}\colon \mathcal{P}_0 \times E_\infty^{\circ\rho} \to E_\infty^{\circ\rho}$ is C^∞. Moreover $U^{(+)}$ is not equivalent by a C^2 map to a linear representation on $E_\infty^{\circ\rho}$: If $U_1^{(+)}$ and $U_2^{(+)}$ are defined via (1.23a) by two choices of the function χ_0, they are equivalent.*

Theorem II. *Let $1/2 < \rho < 1$. There exist an open neighbourhood $\mathcal{U}_\infty$ (resp. $\mathcal{O}_\infty^{(+)}$) of zero in V_∞^ρ (resp. $E_\infty^{\circ\rho}$), a diffeomorphism $\Omega_+\colon \mathcal{O}_\infty^{(+)} \to \mathcal{U}_\infty$ and a C^∞ function $U\colon \mathcal{P}_0 \times \mathcal{U}_\infty \to \mathcal{U}_\infty$, defining a nonlinear representation of $\mathcal{P}_0$, such that:*

i) $\displaystyle \frac{d}{dt} U_{\exp(tX)}(u) = T_X(U_{\exp(tX)}(u)), \quad X \in \mathfrak{p}, t \in \mathbb{R}, u \in \mathcal{U}_\infty,$

ii) $\Omega_+ \circ U_g^{(+)} = U_g \circ \Omega_+$

iii) $$\lim_{t\to\infty} \Big(\|U^M_{\exp(tP_0)}(\Omega_+(u)) - U^{M1}_{\exp(tP_0)}(f, \dot f)\|_{M^\rho} + \|U^D_{\exp(tP_0)}(\Omega_+(u)) - \sum_{\varepsilon=\pm} e^{is_\varepsilon^{(+)}(u,t,-i\partial)} P_\varepsilon(-i\partial) U^{D1}_{\exp(tP_0)}\alpha\|_D \Big) = 0,$$

for $u = (f, \dot f, \alpha) \in \mathcal{O}_\infty^{(+)}$.

This theorem solves in particular *the Cauchy problem* for small initial data and proves *asymptotic completeness.* By the construction of the wave operator Ω_+ the solution $(A(t,\cdot), \dot A(t,\cdot), \psi(t,\cdot)) = U_{\exp(tP_0)}(u)$ of the Cauchy problem satisfies

$$\sup_{\substack{x\in\mathbb{R}^3\\ t\geq 0}} \Big((1+|x|+t)^{\frac{3}{2}-\rho} |A_\mu(t,x)| + (1+|x|+t)|\partial_\nu A_\mu(t,x)| + (1+|x|+t)^{\frac{3}{2}} |\psi(t,x)| \Big) < \infty.$$

Cohomology. These results and conditions (1.14) and (1.17c) have a natural *cohomological interpretation.* A necessary condition for $U^{(+)}$ and Ω_+ to be a solution of equation (1.14) is that the formal power series development of $U_g^{(+)}$, U_g and Ω_+ in the initial conditions satisfy cohomological equations. In particular the second order terms $U_g^{(+)2}$, U_g^2 and Ω_+^2 must satisfy

$$\delta\Omega_+^2 = C^2, \quad \delta R^{(+)2} = 0, \tag{1.24}$$

where δ is the coboundary operator defined by the representation $(g, A) \mapsto U_g^1 A^2(\otimes^2 U_{g-1})$ of the Poincaré group $\mathcal{P}_0$ on bilinear symmetric maps A^2 from $E_\infty^\rho \to E_\infty^\rho$ and where $C^2 = R^{(+)2} - R^2$ is the cocycle defined by $R_g^2 = U_g^2(U_{g-1}^1 \otimes U_{g-1}^1)$ and $R_g^{(+)2} = U_g^{(+)2}(U_{g-1}^1 \otimes U_{g-1}^1)$, $g \in \mathcal{P}_0$. Equation (1.24) shows that the cochain $R^{(+)2}$ has to be a cocycle and then that the cocycle C^2 has to be a coboundary. This is equivalent to the existence of a solution $U^{(+)}$, Ω_+ of equation (1.14) modulo terms of order at least three. There are similar equations for higher order terms:

$$\delta\Omega_+^n = C^n, \quad \delta R^{(+)n} = 0, \tag{1.25}$$

where C^n and $R^{(+)n}$ are functions of $\Omega_+^2, \ldots, \Omega_+^{n-1}$ and $U^{(+)2}, \ldots, U^{(+)n}$.

There exist a modified wave operator and global solutions of the M-D equations for a set of scattering data $(f, \dot{f}, \alpha)$, satisfying $\hat{f}_\mu(k) = \hat{\dot{f}}_\mu(k) = 0$ for k in a neighbourhood of zero, i.e. f_μ and $\dot{f}_\mu$ have no low frequencies component. It follows that the usual wave operator (i.e. $U^{(+)} = U^1$, $s_\varepsilon^{(+)} = 0$ in (1.17c)) does not exist, even in this case where there are no low frequencies. In fact there is an obstruction to the existence of such a solution of equation (1.14) for $n = 3$ due to the self-coupling of ψ with the electromagnetic potential created by the current $\overline{\psi}\gamma_\mu\psi$. However there exists a modified wave operator satisfying (1.25) for $n \geq 2$, with $U^{(+)} = U^1$. Moreover, as we have already pointed out, the phase function $s_\varepsilon^{(+)}$ is given by formula (1.18).

Thus in the absence of low frequencies in the scattering data, for the electromagnetic potential, we can choose Ω_+ such that it intertwines the linear representation U^1 and the nonlinear representation U of $\mathcal{P}_0$. Now if $(f, \dot{f}) \in M_\infty^{\circ\rho}$, $1/2 < \rho < 1$, have nontrivial low frequency part (as for the Coulomb potential ($\hat{f}_\mu(k) \simeq |k|^{-2}$) and necessary to have asymptotic completeness) then there is a *cohomological obstruction* already for $n = 2$ if we want to obtain $U^{(+)2} = 0$. We show that the cocycle R^2 can be split into a trivializable part $-C^2$ (a coboundary) and a nontrivializable part $R^{(+)2}$, which defines $U^{(+)2}$ and therefore the whole representation $U^{(+)}$. We call this nontrivial part the *infrared cocycle.*

$R_g^{(+)2} = 0$ for $g \in SL(2,\mathbb{C})$, $R_g^{(+)2} = (R_g^{(+)2M}, R_g^{(+)2D})$, $R^{(+)2M} = 0$ and

$$\begin{aligned}
&(R_g^{(+)2D}(u_1 \otimes u_2))^\wedge(k) \qquad (1.26)\\
&\quad = \frac{i}{2}\sum_{\varepsilon=\pm}\big(\vartheta^\infty(B_1^{(+)1}(\cdot) - B_1^{(+)1}(\cdot - a), (\omega(k), -\varepsilon k)\big)P_\varepsilon(k)\hat{\alpha}_2(k)\\
&\qquad\qquad + \vartheta^\infty\big(B_2^{(+)1}(\cdot) - B_2^{(+)1}(\cdot - a), (\omega(k), -\varepsilon k)\big)P_\varepsilon(k)\hat{\alpha}_1(k)),
\end{aligned}$$

for $g = \exp(a^\mu P_\mu)$, $u_i = (f_i, \dot{f}_i, \alpha_i) \in E_\infty^{\circ\rho}$, $i \in \{1,2\}$, $B_{i\mu}^{(+)1}$ being the corresponding free field given by (1.22b).

If the limit $\vartheta(B^{(+)1}, (t, -t\varepsilon k/\omega(k)) \to \vartheta^\infty(B^{(+)1}, (\omega(k), -\varepsilon k))$ exists for $t \to \infty$, the infrared cocycle is a coboundary. In particular, this is the case when $\hat{f}_\mu, \hat{\dot{f}}_\mu$ are equal to zero in a neighbourhood of zero.

Physical Remarks

The asymptotic condition in statement iii) of Theorem II is equivalent to (when $t \to \infty$)

$$\|(A(t), \dot{A}(t)) - (A_L(t), \dot{A}_L(t))\|_{M^\rho} + \|\psi(t) - (e^{-i\varphi}\psi_L)(t)\|_D \to 0, \quad (1.27)$$

$(A(t), \dot{A}(t), \psi(t)) = U_{\exp(tP_0)}(\Omega_+(u))$, $(A_L(t), \dot{A}_L(t), \psi_L(t)) = U^1_{\exp(tP_0)}u$ and for a fixed $u = (f, \dot{f}, \alpha) \in \mathcal{O}_\infty^{(+)}$, φ is some function in $C^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$. Therefore we have $A_\mu \to A_\mu^{\text{free}}$ and $\psi \to e^{-i\varphi}\psi^{\text{free}}$.

The *asymptotic representation* $U^{(+)}$ *is not linear*: it is equal to U^1 modulo a gauge map $Q_g = U_g^{(+)}U^1_{g^{-1}}$. But the energy-momentum tensor for the M–D system, given by

$$t^{\mu\nu} = -F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}(\overline{\psi}\gamma^\mu(i\partial^\nu - A^\nu)\psi + (\overline{(i\partial^\nu - A^\nu)\psi})\gamma^\mu\psi),$$

where $0 \le \mu \le 3$, $0 \le \nu \le 3$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; and the current density vector $j^\mu = \overline{\psi}\gamma^\mu\psi$ are invariant under gauge transformations $\psi' = e^{i\lambda}\psi$, $A'_\mu = A_\mu - \partial_\mu\lambda$ (not necessarily respecting the Lorentz gauge condition that gives here $\Box\lambda = 0$). Therefore they go to their free limits when $t \to \infty$. Condition (1.17c) is therefore natural. A similar discussion can be made for the relativistic angular momentum tensor.