Scaling Lee-Yang model on fluctuating sphere

① Liouville field theory

- Action

coupling

(renormalized) Liouville field

$$\mathcal{L}_L = \frac{1}{4\pi}(\partial_a \varphi)^2 + \mu e^{2b\varphi}$$

cosmological constant

$Q = b + b^{-1} \leftarrow$ background charge

- Holomorphic stress tensor

$$T(z) = -(\partial\varphi)^2 + Q\,\partial^2\varphi$$
$$\bar{T}(\bar{z}) = -(\bar{\partial}\varphi)^2 + Q\,\bar{\partial}^2\varphi$$

$c_L = 1 + 6Q^2 \geq 25 \leftarrow$ Liouville central charge

- Exponential operators

$V_\alpha = e^{2\alpha\varphi} \qquad \mathrm{Re}\,\alpha < Q/2$

dimension $\Delta_\alpha = \alpha(Q-\alpha)$

• Two- and three-point functions

$$\langle V_\alpha V_\alpha \rangle = D(\alpha) = \frac{(\pi\mu\gamma(b^2))^{\frac{Q-2\alpha}{b}}}{b^2}\frac{\gamma(2\alpha b - b^2)}{\gamma(2+b^{-2}-2\alpha b^{-1})}$$

$$\gamma(x) = \Gamma(x)/\Gamma(1-x)$$

$$C_3(\alpha_1\alpha_2\alpha_3) = \langle V_{\alpha_1}V_{\alpha_2}V_{\alpha_3}\rangle = \left[\pi\mu\gamma(b^2)\,b^{2-2b^2}\right]^{\frac{Q-\Sigma\alpha}{b}} \times$$

$$\frac{\Upsilon(b)\,\Upsilon(2\alpha_1)\,\Upsilon(2\alpha_2)\,\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\,\Upsilon(\alpha_1+\alpha_2-\alpha_3)\,\Upsilon(\alpha_2+\alpha_3-\alpha_1)\,\Upsilon(\alpha_3+\alpha_1-\alpha_2)}$$

Υ - function

$$\Upsilon(x+b) = \gamma(bx)\,b^{1-2bx}\,\Upsilon(x)$$
$$\Upsilon(x+b^{-1}) = \gamma(b^{-1}x)\,b^{2b^{-1}x-1}\,\Upsilon(x)$$
$$\Upsilon(x) = \Upsilon(Q-x)$$

• Four-point function depends on modular parameters $x, \bar{x}$

$$C_4\begin{pmatrix}\alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4\end{pmatrix} x\,\bar{x}\Big) = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4}\rangle$$

$$= \frac{1}{2}\int_P C_3\!\left(\alpha_1\alpha_2\tfrac{Q}{2}+iP\right) C_3\!\left(\tfrac{Q}{2}-iP,\alpha_3\alpha_4\right)\left|\mathcal{F}\begin{pmatrix}\alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4\end{pmatrix} P \,\Big|\, x\Big)\right|^2$$

4-point conformal block

II CFT coupled to Liouville

- Formal action

$$A_{gr} = \int\left(\frac{1}{4\pi}(\partial\varphi)^2 + \mu e^{2b\varphi}\right) + A_{CFT} + \lambda\int \Phi_\Delta e^{2g\varphi} + A_{sp}$$

Liouville; CFT of cent. ch. c; perturb. of dim. Δ; spectator CFT

Total matter central charge

$$c_M = c + c_{sp}$$

- Gravitational balance

$c_L + c_M = 26$ ← central charge balance

$\Delta_g + \Delta = 1$ ← „Dressing" balance

$$\left.\begin{aligned} b &= \sqrt{\frac{25-c_M}{24}} - \sqrt{\frac{1-c_M}{24}} \\ g &= \sqrt{\frac{25-c_M}{24}} - \sqrt{\frac{1-c_M}{24} + \Delta} \end{aligned}\right\} \text{KPZ scaling formulas}$$

- Sphere partition function $Z(\mu,\lambda)$

$$\left(b\mu\frac{\partial}{\partial\mu} + \lambda g\frac{\partial}{\partial\lambda} + Q\right) Z(\mu,\lambda) = 0$$

$$Z(\mu,\lambda) = \mu^{Q/b} F\left(\frac{\lambda}{\mu^s}\right) \qquad \boxed{s = g/b}$$

- Fixed area partition function

Area $A = \int e^{2b\varphi}$

$$Z(\mu,\lambda) = \int_0^\infty Z(A,\lambda) e^{-\mu A} \frac{dA}{A}$$

III Perturbative ($A \to 0$) expansion

- $$Z(\mu,\lambda) = \underbrace{Z(\mu,0)}_{Z_L(\mu)\, Z_{CFT}\, Z_{sp}} \left\langle e^{-\lambda \int \Phi_A e^{2g\varphi}} \right\rangle$$

$$\frac{Z(\mu,\lambda)}{Z(\mu,0)} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int \langle \Phi_1 \cdots \Phi_n \rangle_{CFT} \langle e^{2g\varphi_1} \cdots e^{2g\varphi_n} \rangle_{Liou}$$

contains factor μ^{-ns}

$$= \sum_n a_n \left(-\frac{\lambda}{\mu^s}\right)^n$$

- Fixed area expansion

$$\frac{Z(A,\lambda)}{Z(A,0)} = z(h) = \sum_n z_n (-h)^n$$

dimensionless $h = \lambda \left(\frac{A}{\pi}\right)^s$

$$z_n = a_n \frac{\pi^{ns}\, \Gamma(-Q/b)}{\Gamma(ns - Q/b)}$$

- In explicit CFT constructions $\langle \Phi_1 \cdots \Phi_n \rangle$ is constructed in principle (effectively up to 4-points).
  LFT correlation functions $\langle e^{2g\varphi_1} \cdots e^{2g\varphi_n} \rangle$ are known explicitly up to 3 points.
  4-point is effectively computable

(IV) Criticality (large A behavior)

- Exact solutions of matrix models show that generally $\sum a_n(-\lambda/\mu^s)^n$ has finite radius of convergence limited by a critical singular point $\lambda = \lambda_c(\mu)$. It corresponds to critical behavior when the surface grows large $A \to \infty$.

For fixed area partition function

$$Z(A,\lambda) \sim A^{Q'/b'} e^{-\mu_c(\lambda) A}$$

specific energy induced by noncritical matter

$$\mu_c = -f_0 \lambda^{1/s}$$

dimensionless number

- IR scaling parameters b', Q'

Typically perturbed matter develops finite correlation length $\sim \lambda^{1/2s}$ and at $A \to \infty$

$$c_M \to c_M - c = c_{sp}$$

$$b' = \sqrt{\frac{Q^2}{4} + \frac{c}{24}} - \sqrt{\frac{Q^2}{4} + \frac{c}{24} - 1}$$

$$Q' = b' + \frac{1}{b'}$$

(V) Scaling Lee-Yang model

- $\mathcal{M}(2/5)$ minimal CFT

  $c = -\frac{22}{5}$ primaries $\begin{cases} I & \Delta = 0 \\ \phi & \Delta = -\frac{1}{5} \end{cases}$

  Basic OPE

  $$\phi\phi = I + C_{\phi\phi}^{\phi}\,\phi$$

  $$C_{\phi\phi}^{\phi} = i x \leftarrow \text{pure imaginary}$$

  $$x = \left(\frac{\sqrt{5}-1}{2}\right)^{1/2} \frac{\Gamma^2(1/5)}{5\,\Gamma(3/5)\,\Gamma(4/5)} = 1.9113\ldots$$

- Perturbed model

  $$A_{LY} = A_{\mathcal{M}(2/5)} + i\lambda \int \phi$$

  is integrable in flat.

  Particle mass $m = k(2\pi\lambda)^{5/12}$

  $$k = \frac{4\sqrt{\pi}}{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}\left[\frac{\Gamma(4/5)\,\Gamma(3/5)}{100\,\Gamma(1/5)\,\Gamma(2/5)}\right]^{5/24}$$

- Exact vacuum energy in flat

  $$\mathcal{E}_{vac} = -\frac{m^2}{4\sqrt{3}}$$

  $\downarrow$

  In classical limit $b^2 \to 0$ (rigid surface)

  $$f_0 = \frac{k^2}{4\sqrt{3}} = 0.2179745\ldots$$

VI Analytic-numeric analysis

- $$\frac{Z(A,\lambda)}{Z(A,0)} = z(h) = \sum_n z_n (-h)^n$$

  entire function of h

  $z_0 = 1$

  $z_1 = 0$

  $\left.\begin{array}{l} z_2 \\ z_3 \\ \text{Sometimes } z_4 \end{array}\right\}$ known numerically

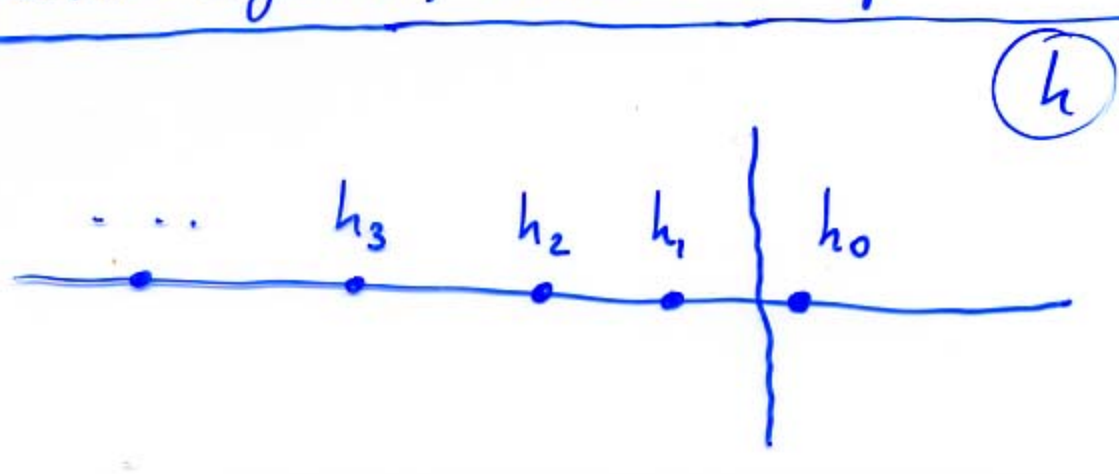
- $h \to \infty$ $(A \to \infty)$ asymptotic

$$\log \frac{Z(A,\lambda)}{Z(A,0)} \sim \pi f_0 h^{4/5} + \frac{1}{5}\left(\frac{Q}{b} - \frac{Q'}{b'}\right)\log h + \ldots$$

less important terms

- Observations

$$\boxed{\begin{array}{l} \text{(1) In this particular model LY+Liouv+SP} \\ \text{(\underline{not in general!}) this asymptotic holds in} \\ \text{the whole plane of } h \ \ |\arg h| < \pi \\ \text{(2) All zeros are real (one positive, the} \\ \text{rest - negative) accumulating to } -\infty \end{array}}$$



- Asymptotic of zeros at $n \to \infty$

$$-h_n \sim \left(\frac{n-\delta}{f_0 \sin\frac{\pi}{s}}\right)^s$$

$$\delta = \frac{Q/b - Q'/b'}{s} - \frac{1}{2}$$

- Sum rules (perturbative information)

$$\left.\begin{aligned} &\sum \frac{1}{h_n} = 0 \\ &\sum \frac{1}{h_n^2} = -2 z_2 \\ &\sum \frac{1}{h_n^3} = 3 z_3 \end{aligned}\right\} \text{Expressed in terms of first perturbative coeff.}$$

...

- Algorithm
  - Approximate $h_N, \dots \infty$ by asymptotic
  - Solve first $N+1$ sum rules for $h_0, h_1 \dots h_{N-1}$ and f_0

In practice $N = 1, 2$, sometimes 3

if z_4 is known

VII Numerical results for LY

- Precision-convergence estimate

– semicl. region ($b^2=0$) 0.08% $N=2$
0.02% $N=3$

– Minimal gravity $b^2=0.4$ (exactly solvable) 1.0% $N=2$
0.25% $N=3$

- Critical b_c

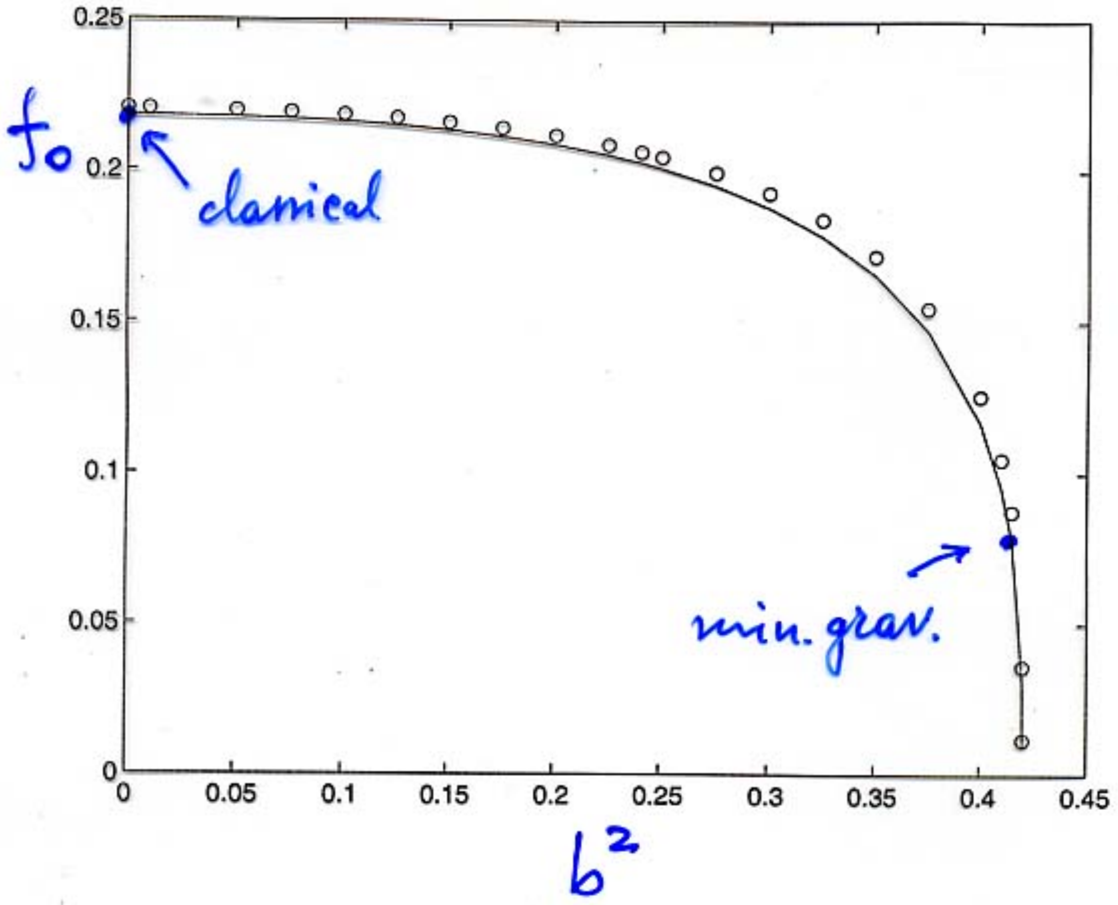
$$b_c^2 = 0.420204\ldots$$ (close to minimal gravity point

$$f_0 \sim G_c (b_c^2 - b^2)^\alpha$$

$$\alpha = \frac{1}{2 s_c} \sim 0.29$$

- Small b^2 behavior

$$f_0(b^2) = \underset{\substack{\uparrow \\ 0.21798..}}{f_0(0)} - b^2 \underset{\substack{\uparrow \\ 0.00466..}}{f_1} - \ldots$$



0.25
0.2
0.15
0.1
0.05
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
f_0
classical
min. grav.
b^2