Fluctuations in Statistical Models

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1). V.V. Begun, M. Gazdzicki, M.I. Gorenstein, O.S. Zozulya,
Phys. Rev. C70, 034901 (2004).

2). V.V. Begun, M.I. Gorenstein, A.P. Kostyuk, O.S. Zozulya,
Phys. Rev. C71, 054904 (2005).

3). V.V. Begun, M.I. Gorenstein, O.S. Zozulya,
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4). A. Keranen, F. Becattini, V.V. Begun, M.I. Gorenstein, O.S. Zozulya,
J.Phys. G31 S1095-S1100 (2005).

5). V.V. Begun, M.I. Gorenstein, A.P. Kostyuk, O.S. Zozulya,
nucl-th/0505069.

6). F. Becattini, et al., nucl-th/0507039

Statistical ensembles:

Microcanonical – **E**, **V**, **Q**

Canonical – **T**, **V**, **Q**

Grand Canonical – **T**, **V**, μ_Q

Grand Microcanonical – **E**, **V**, μ_Q

Fluctuations

$$\omega^i \equiv \frac{\langle N_i^2 \rangle - \langle N_i \rangle^2}{\langle N_i \rangle}$$

$$N_i = N_+ \, , \; N_- \, , \; N_{ch} = N_+ + N_- \, , \; N_{\pi^+} \, , N_{\pi^-} \, , \; N_{\overline{p}} \, , \; ...$$

$$P(N) = \frac{\overline{N}^N}{N!} \exp(-\overline{N})$$

$$\langle N \rangle = \sum_{N=0}^{\infty} N \; P(N) = \overline{N}$$

$$\langle N^2 \rangle = \sum_{N=0}^{\infty} N^2 \; P(N) = \overline{N}^2 + \overline{N}$$

$$\omega = \frac{\overline{N}^2 + \overline{N} - \overline{N}^2}{\overline{N}} = 1$$

Partition Function in GCE:

$$Z_{g.c.e.}(V,T,\mu) = \sum_{N_{1+},\, N_{1-}=0}^{\infty} \ldots \sum_{N_{j+},\, N_{j-}=0}^{\infty} \ldots$$
$$\times \frac{\left(\lambda_{1+} z_1\right)^{N_{1+}}}{N_{1+}!} \frac{\left(\lambda_{1-} z_1\right)^{N_{1-}}}{N_{1-}!} \ldots \frac{\left(\lambda_{j+} z_j\right)^{N_{j+}}}{N_{j+}!} \frac{\left(\lambda_{j-} z_j\right)^{N_{j-}}}{N_{j-}!} \ldots$$
$$= \prod_j \exp\left(\lambda_{j+} z_j + \lambda_{j-} z_j\right) = \exp\left[\, 2z \, \cosh\left(\frac{\mu}{T}\right)\right] ,$$

where j numerates the spesies, $\lambda_{j\pm} = \exp(\pm\mu/T)$, z_j is a single particle partition function

$$z_j = \frac{g_j V}{2\pi^2} \int_0^\infty k^2 dk \, \exp\left[-\frac{(k^2+m_j^2)^{1/2}}{T}\right] = \frac{g_j V}{2\pi^2} \, T\, m_j^2 \, K_2\left(\frac{m_j}{T}\right) ,$$

and $z \equiv \sum_j z_j$.

$$V \to \infty \quad \Rightarrow \quad z \to \infty$$

Partition Function in CE:

$$
\begin{aligned}
Z_{c.e.}(V,T,Q) \;=\; & \sum_{N_{1+},\, N_{1-}=0}^{\infty} \ldots \sum_{N_{j+},\, N_{j-}=0}^{\infty} \ldots \\
& \times \frac{\left(\lambda_{1+} z_1\right)^{N_{1+}}}{N_{1+}!} \; \frac{\left(\lambda_{1-} z_1\right)^{N_{1-}}}{N_{1-}!} \; \ldots \frac{\left(\lambda_{j+} z_j\right)^{N_{j+}}}{N_{j+}!} \frac{\left(\lambda_{j-} z_j\right)^{N_{j-}}}{N_{j-}!} \ldots \\
& \times \delta\left[\left(N_{1+} + \ldots + N_{j+} + \ldots - N_{1-} - \ldots - N_{j-} - \ldots\right) - Q\right] \\
= \; & I_Q(2z) \; .
\end{aligned}
$$

The microcanonical partition function (m=0):

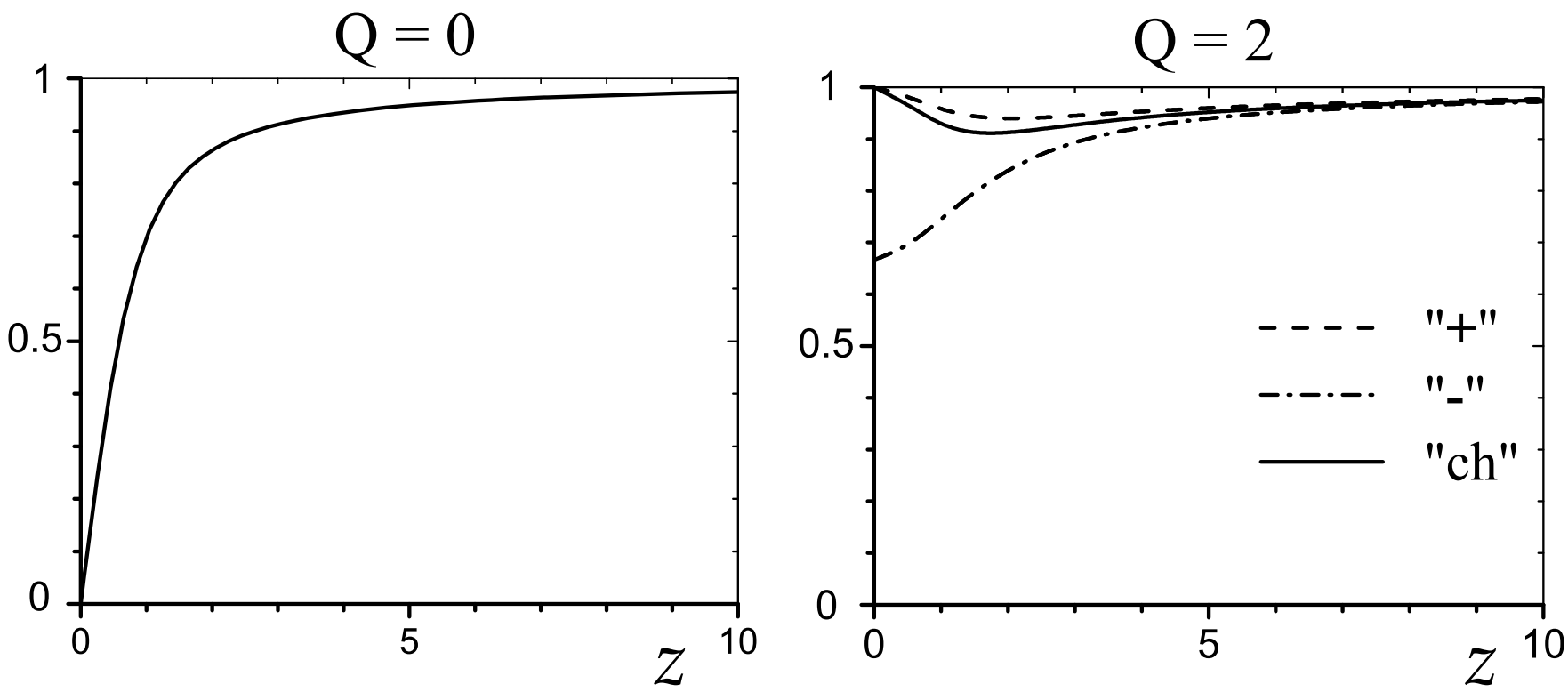
$$W_N(E,V) = \frac{1}{N!}\frac{gV}{(2\pi)^3}\int d^3p^{(N)} \ldots \frac{gV}{(2\pi)^3}\int d^3p^{(1)}\,\delta(E-\sum_{k=1}^{N}|\vec{p}^{\,(k)}|)$$
$$= \frac{1}{E}\frac{x^N}{(3N-1)!N!}\ ,$$

$$W(E,V) \equiv \sum_{N=1}^{\infty} W_N(E,V) = \frac{x}{2E}\ {}_0F_3\left(;\frac{4}{3},\frac{5}{3},2;\frac{x}{27}\right)\ ,$$

where $x \equiv gVE^3/\pi^2$. and $\langle N\rangle_{g.c.e.} \equiv \overline{N} = \left(\frac{x}{27}\right)^{1/4}$.

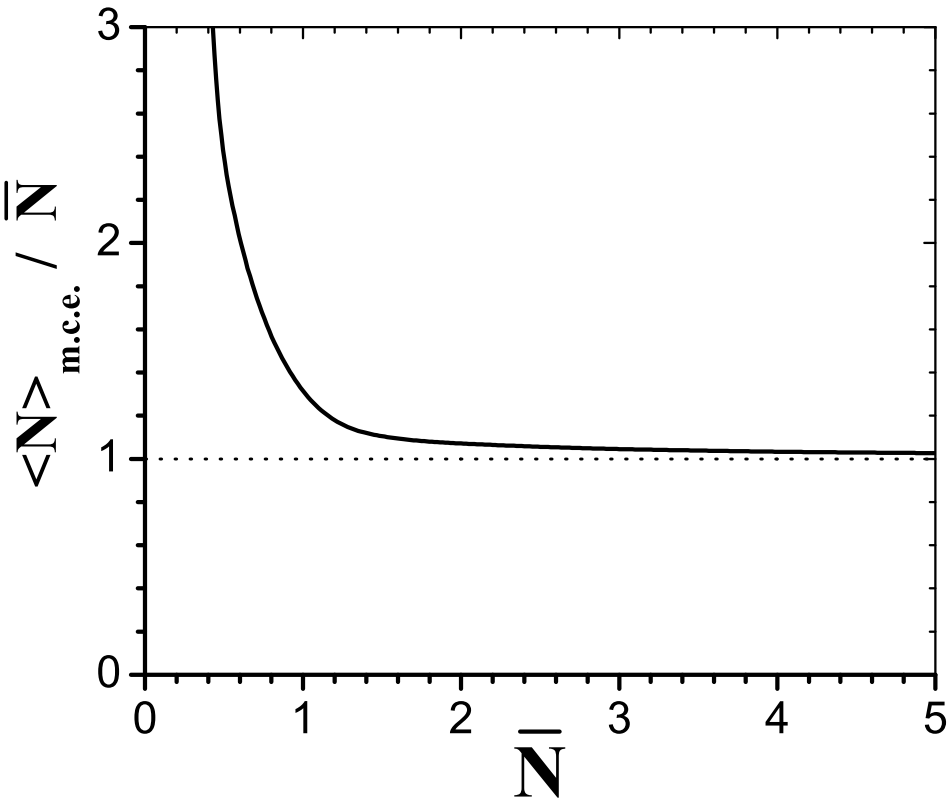
CE/GCE ratio

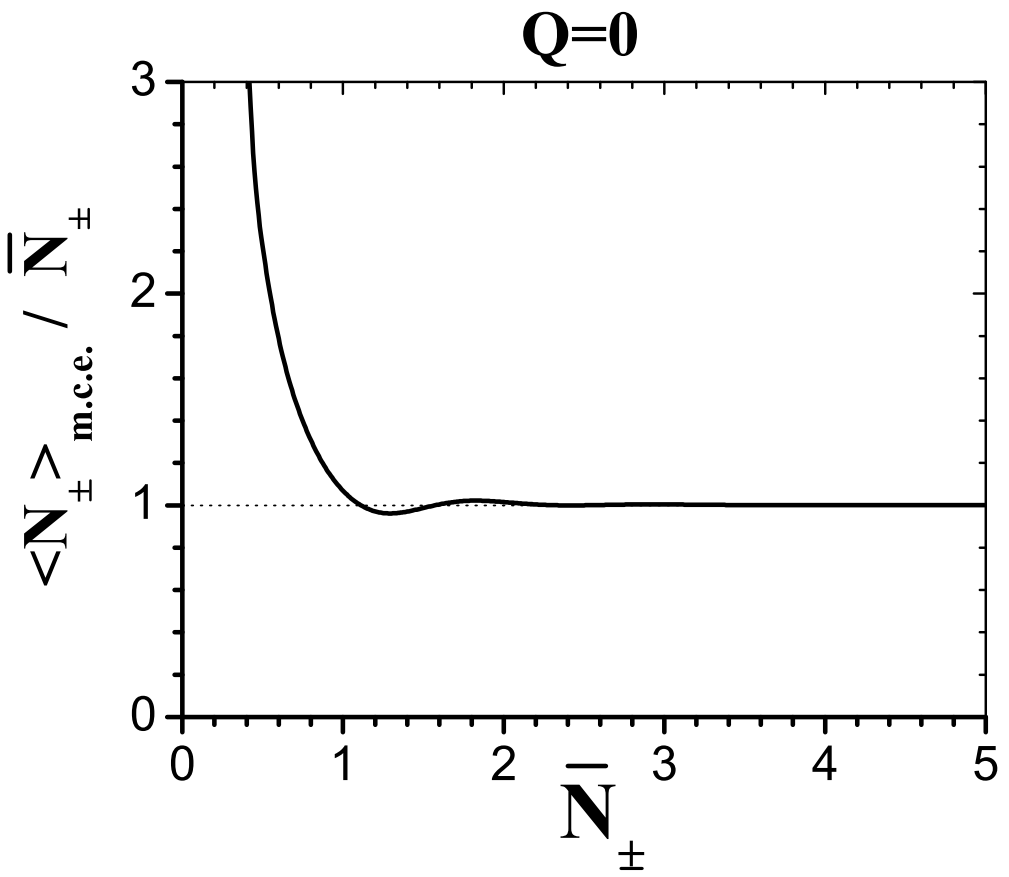
$\frac{\langle N_{\pm} \rangle_{c.e.}}{\langle N_{\pm} \rangle_{g.c.e.}}$, $\frac{\langle N_{ch} \rangle_{c.e.}}{\langle N_{ch} \rangle_{g.c.e.}}$



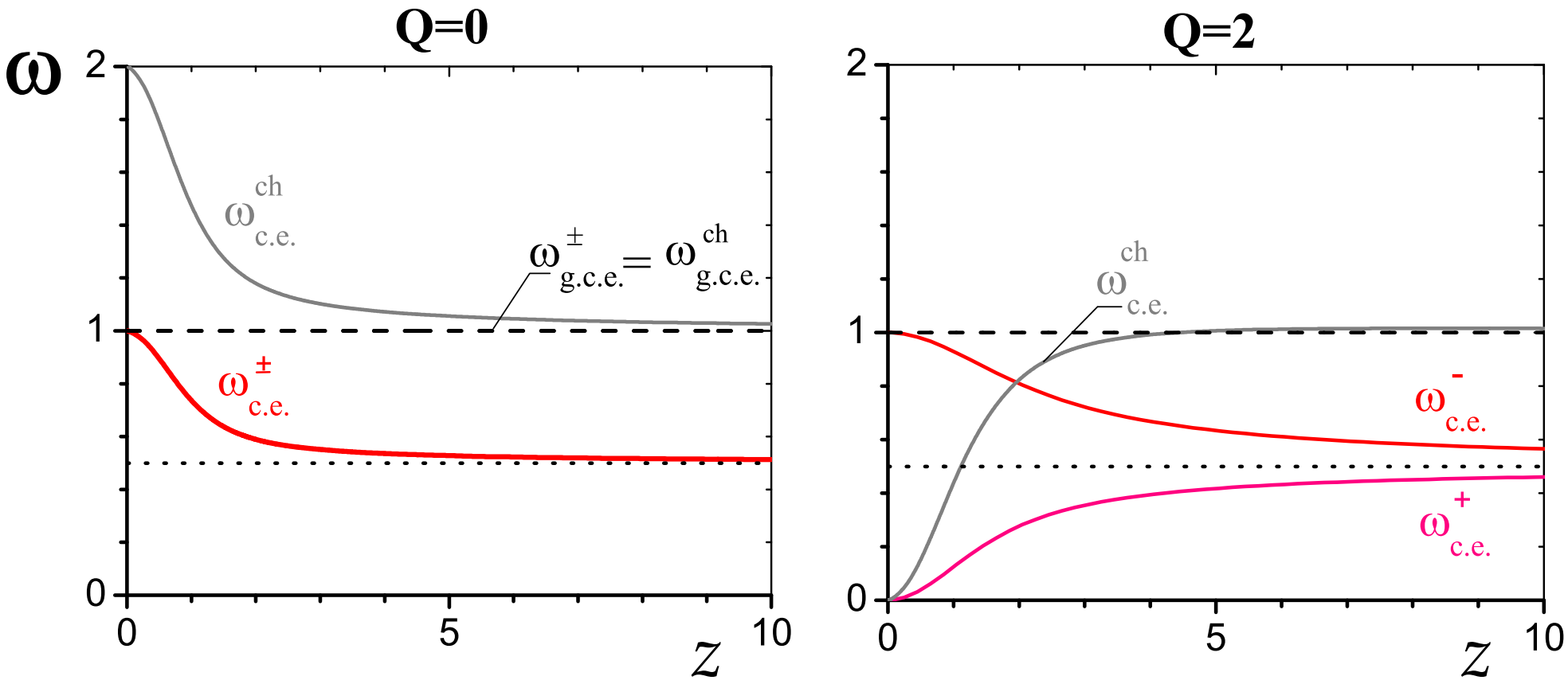
where $\langle N_{j\pm} \rangle = a_{\pm}\ z_j$, and $a_{\pm}^{g.c.e.} = \exp\left(\pm\frac{\mu}{T}\right)$, $a_{\pm}^{c.e.} = \frac{I_{Q\mp1}(2z)}{I_Q(2z)}$,

MCE/GCE ratio (m=0)

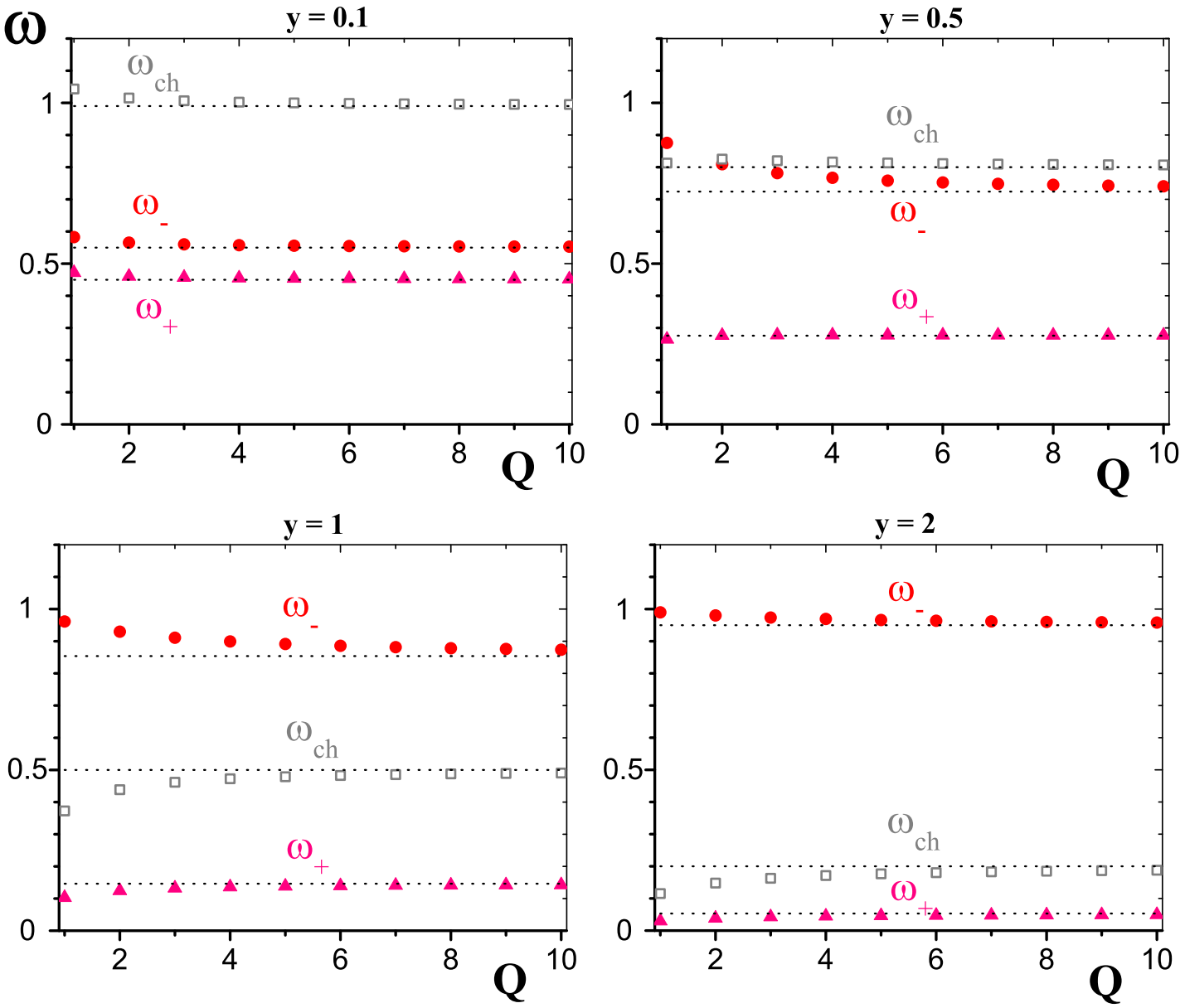




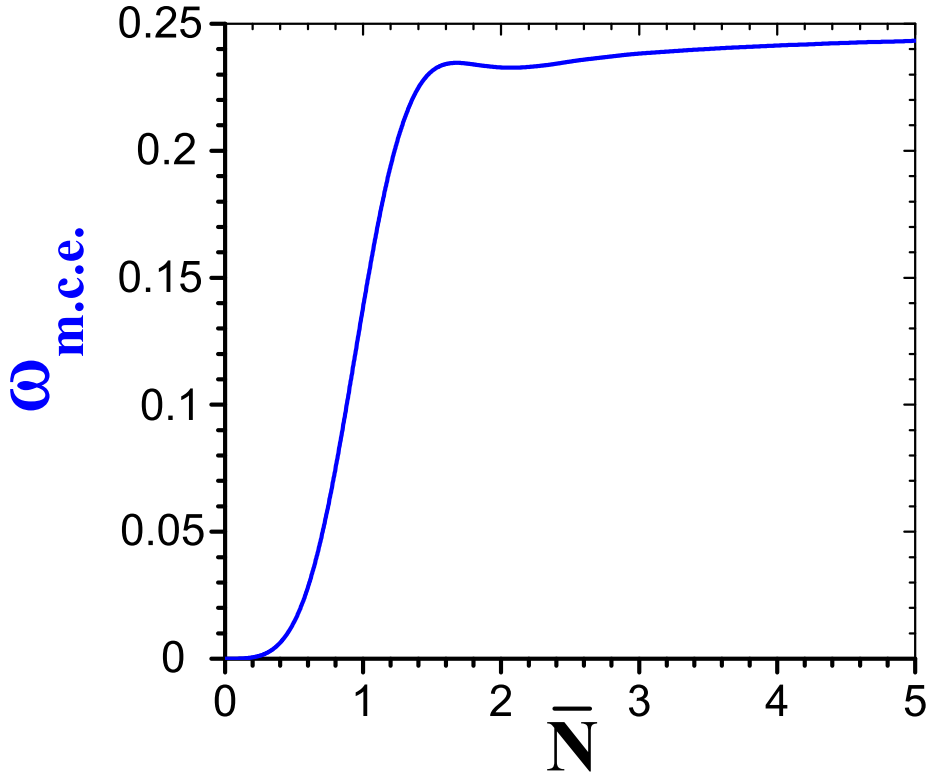
CE Fluctuations

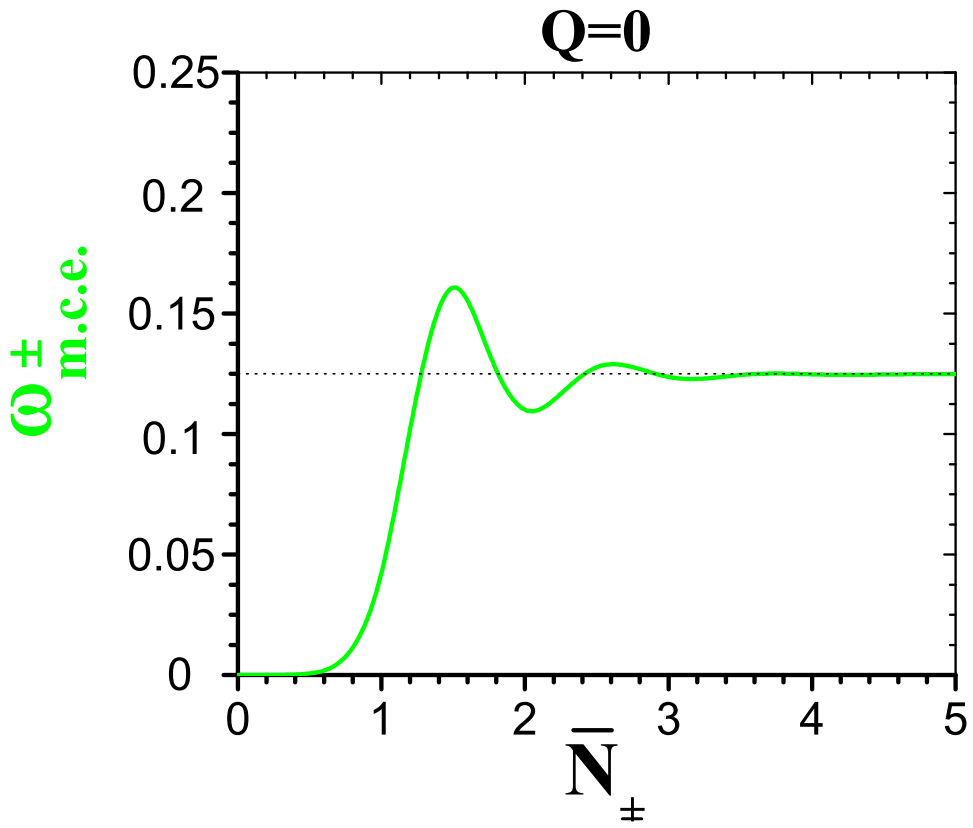


CE Fluctuations ($Q \neq 0$)

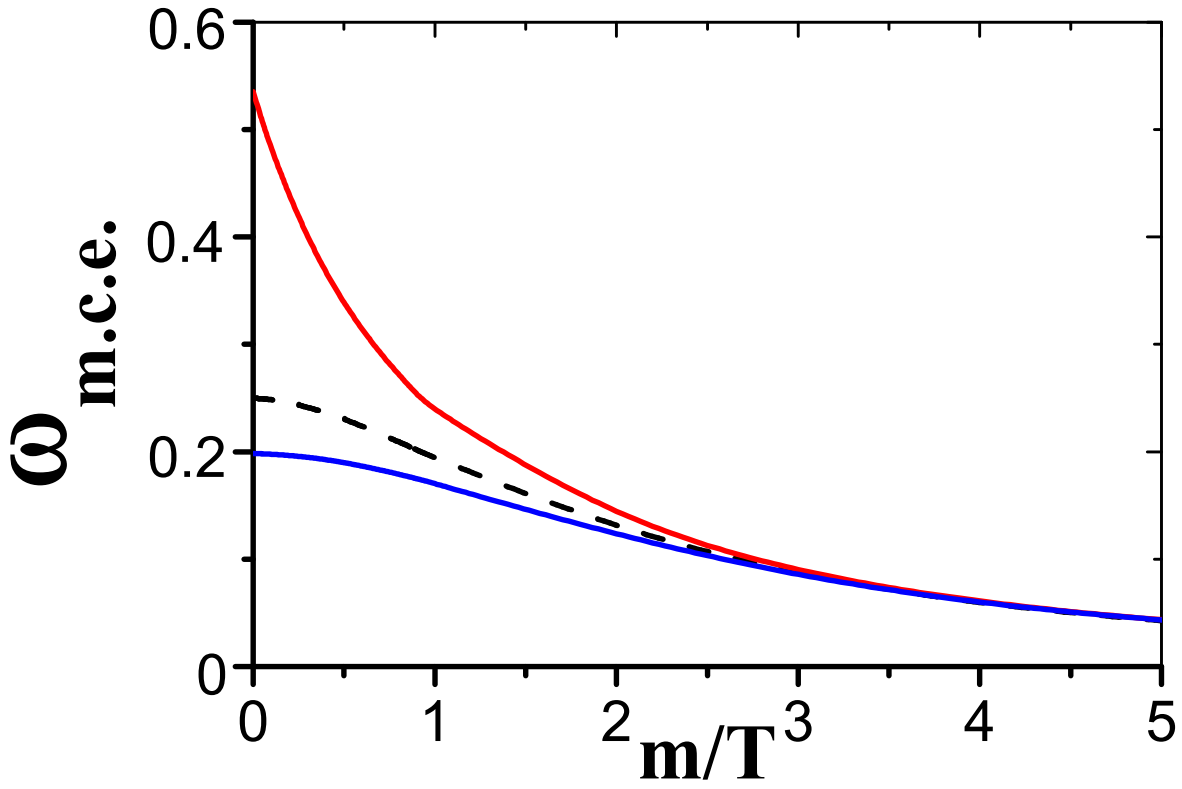


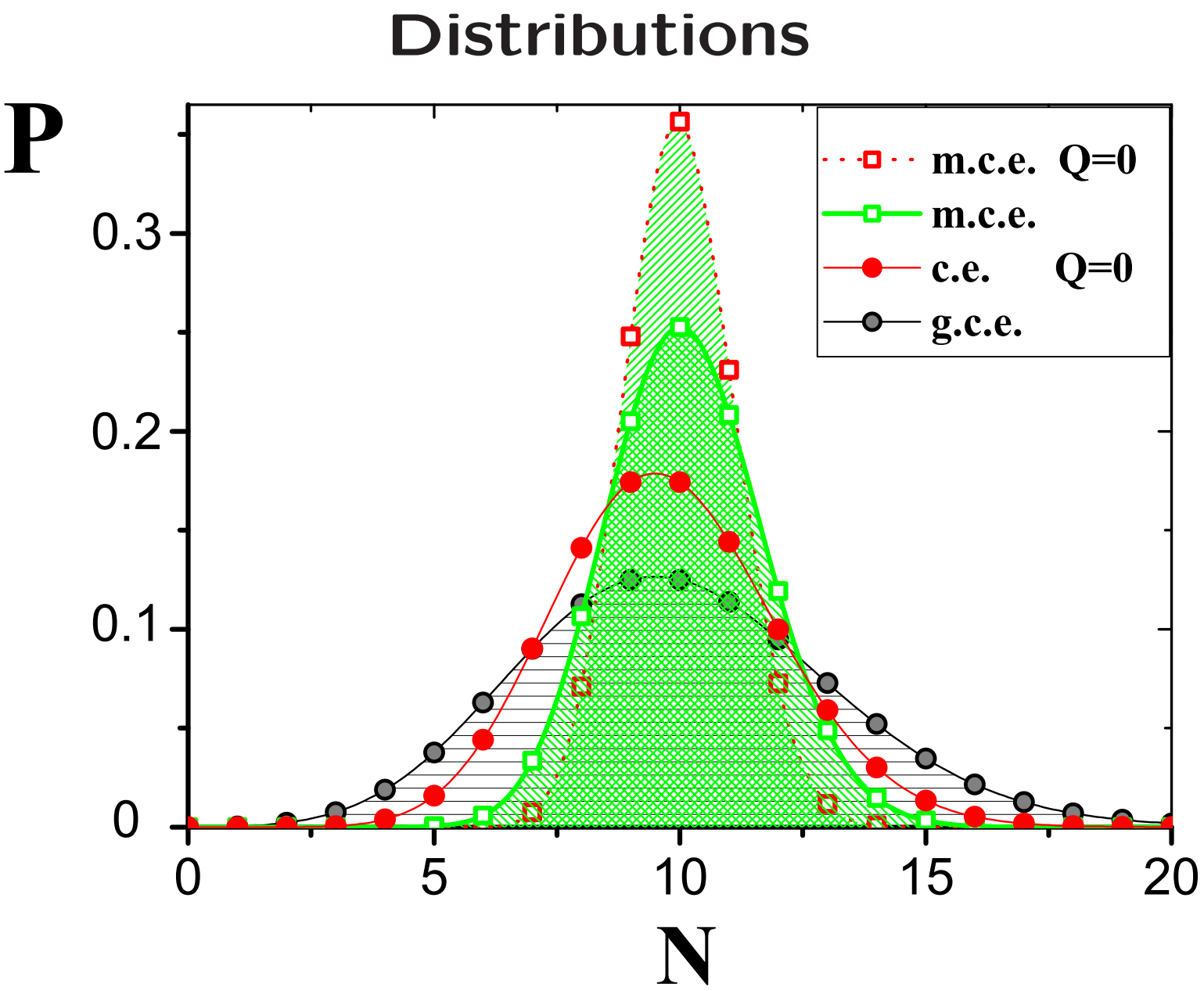
MCE Fluctuations (m=0)





MCE Fluctuations ($m \neq 0$)




Distributions
P
0.3
0.2
0.1
0
0
5
10
15
20
N
m.c.e. Q=0
m.c.e.
c.e. Q=0
g.c.e.

Bose and Fermi Statistics

$$\langle n_p^{\pm} \rangle_{g.c.e.} = \frac{1}{\exp\left[\left(\sqrt{p^2+m^2} \mp \mu\right)/T\right] - \gamma}$$

$$Q \equiv \langle N_+ \rangle_{g.c.e.} - \langle N_- \rangle_{g.c.e.} = \sum_p \langle n_p^+ \rangle_{g.c.e.} - \sum_p \langle n_p^- \rangle_{g.c.e.}$$

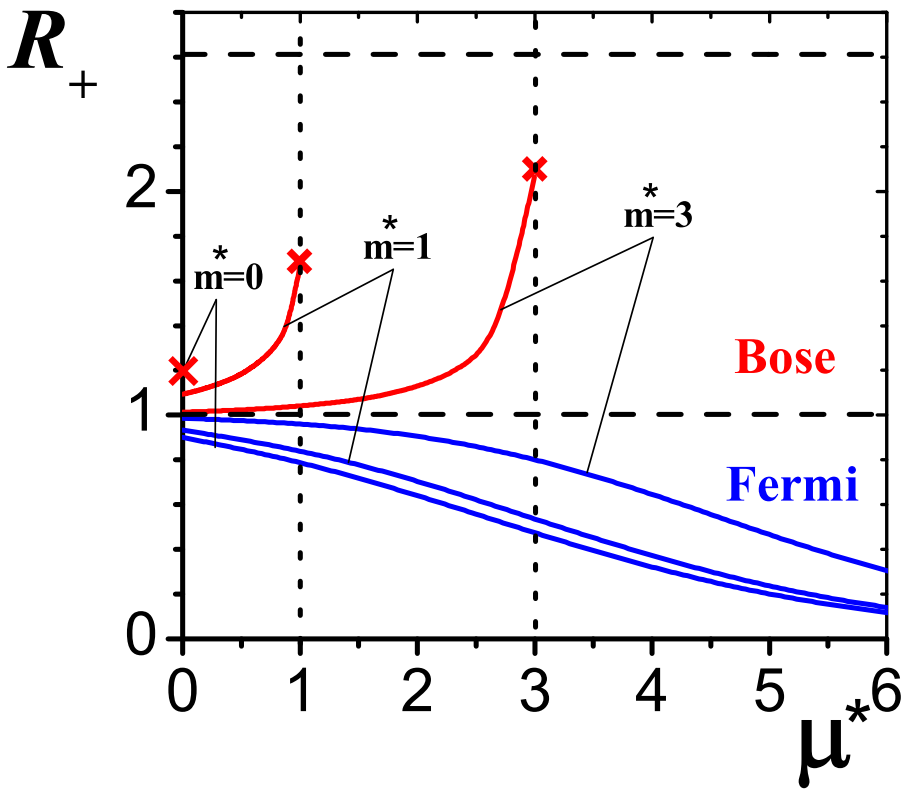
$$\sum_p \ldots = \frac{gV}{2\pi^2} \int_0^\infty p^2 dp \ \ldots$$

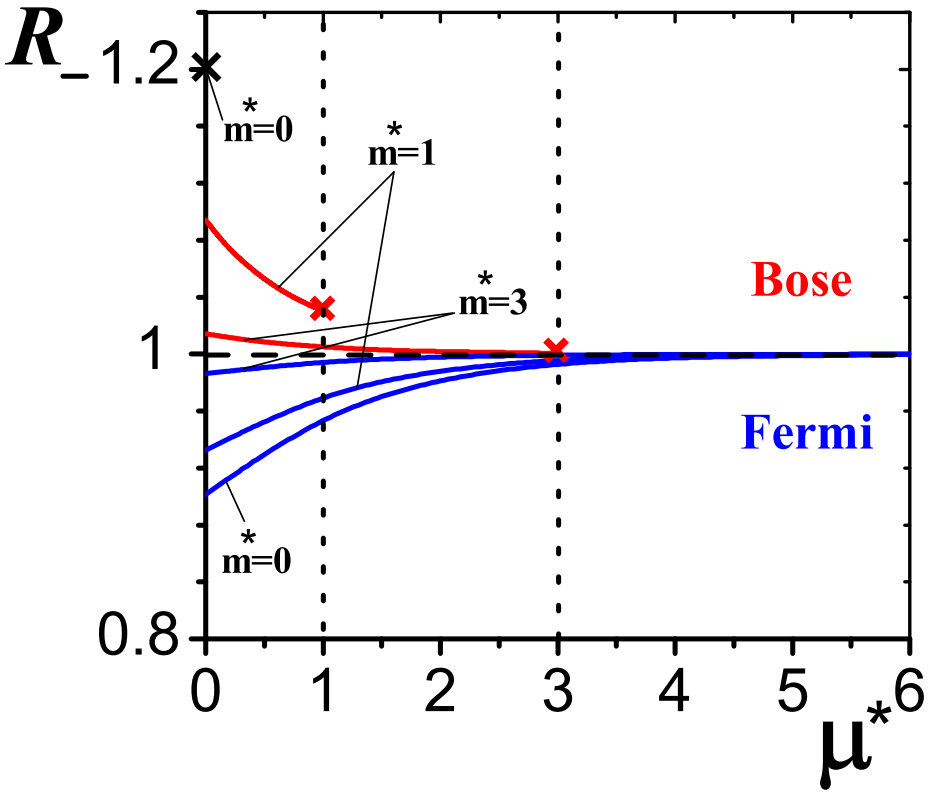
.

$$\rho_\pm \equiv \frac{\langle N_\pm \rangle_{g.c.e.}}{V} = \frac{\sum_p \langle n_p^\pm \rangle_{g.c.e.}}{V} = \frac{g}{2\pi^2} \int_0^\infty \frac{p^2 dp}{\exp\left[\left(\sqrt{p^2+m^2} \mp \mu\right)/T\right] - \gamma}$$

$$= \frac{g\,T^3}{2\pi^2} \int_0^\infty \frac{x^2 dx}{\exp\left[\sqrt{x^2+m^{*\,2}} \mp \mu^*\right] - \gamma} \,,$$

$$m^* \equiv m/T, \ \mu^* \equiv m/T \ \rho_Q \equiv \rho_+ - \rho_- \geq 0, \text{ i.e. } \mu \geq 0$$





$$R_{\pm}^{Bose} \equiv \frac{\rho_{\pm}^{Bose}}{\rho_{\pm}^{Boltz}}, \qquad R_{\pm}^{Fermi} \equiv \frac{\rho_{\pm}^{Fermi}}{\rho_{\pm}^{Boltz}},$$

Bose Condensation

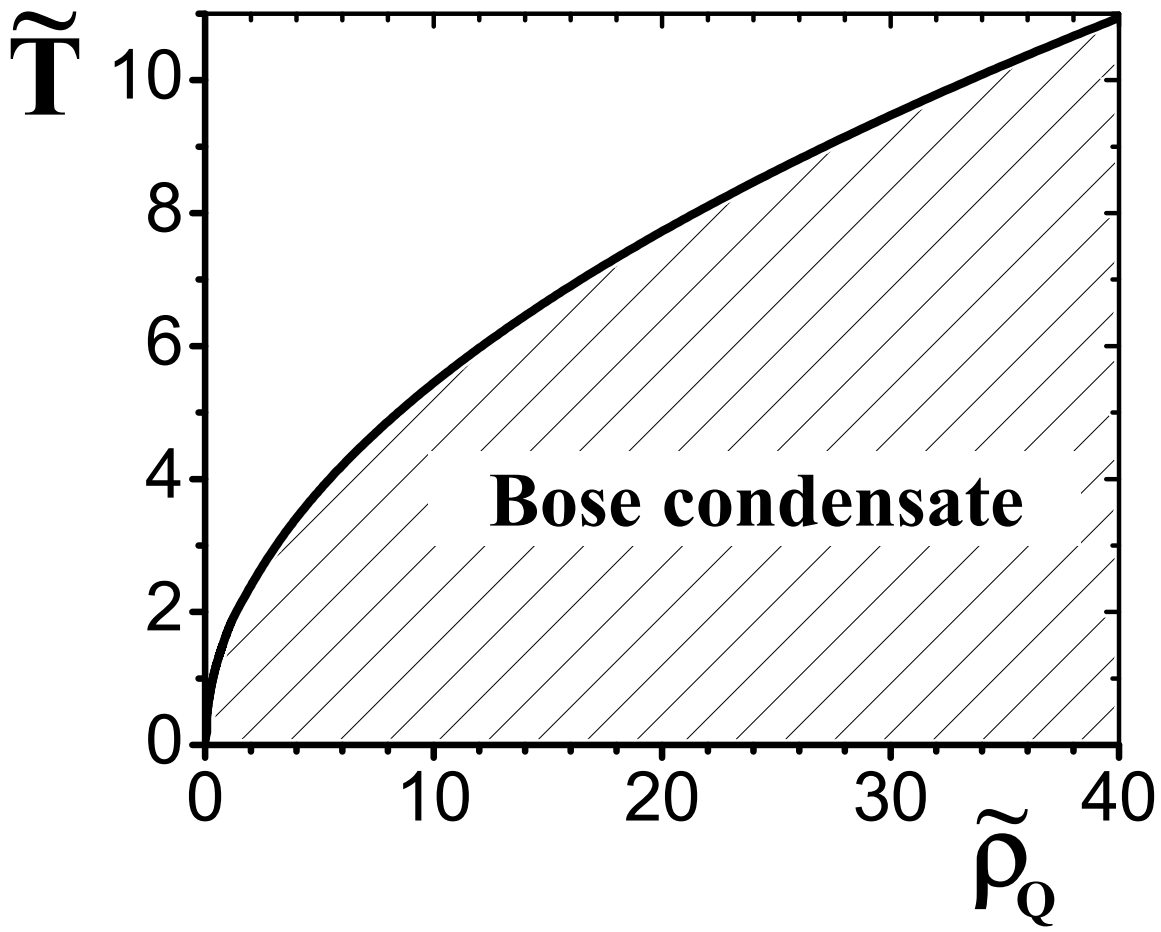
$$\tilde{\rho}_Q \equiv \frac{\rho_Q}{g m^3} \equiv \frac{\rho_+}{g\, m^3} - \frac{\rho_+}{g\, m^3}$$
$$= \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n\, m^*} K_2\left(n\, m^*\right) \sinh(n\, \mu^*) \ .$$

The Bose condensation starts at the point $T = T_C$ when $\mu = \mu^{max} = m$

$$\tilde{\rho}_Q = \frac{\tilde{T}_C}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} K_2\left(n/\tilde{T}_C\right) \sinh(n/\tilde{T}_C) \ ,$$

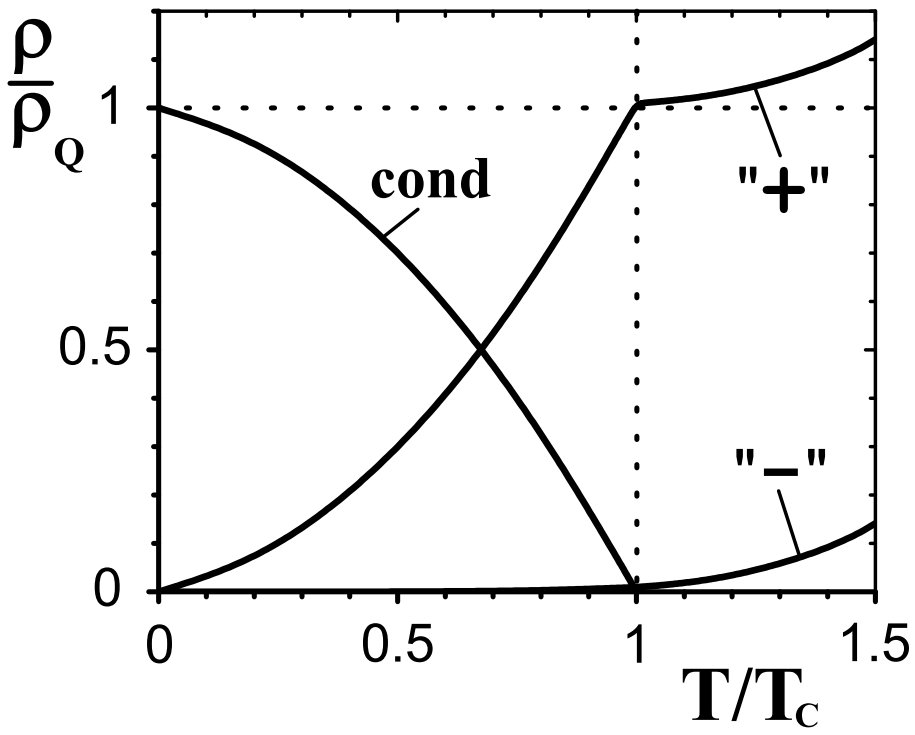
$$\tilde{T}_C \equiv T_C/m$$

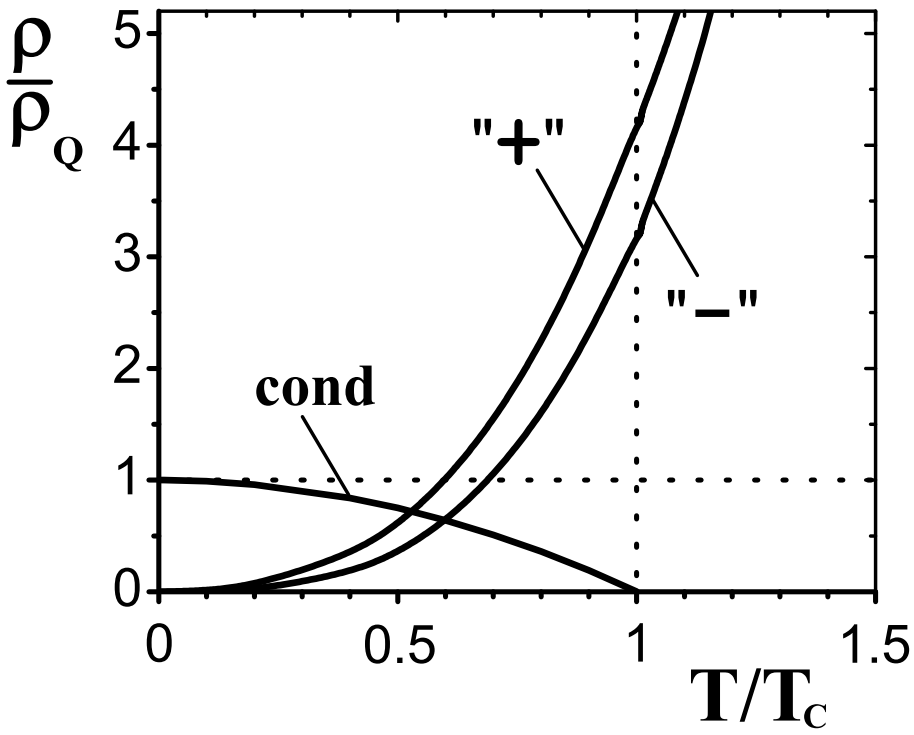
.



$$\tilde{T}_C \simeq 3.313\ \tilde{\rho}_Q^{2/3}\ , \qquad \tilde{T}_C \ll 1$$

$$\tilde{T}_C \simeq 1.732\ \tilde{\rho}_Q^{1/2}\ , \qquad \tilde{T}_C \gg 1$$





At $T < T_C$:

$$\rho_Q = \rho_+(T, \mu = m) - \rho_-(T, \mu = m) + \rho_+^{cond}(T)$$

Particle number fluctuations in the GCE

$$\langle \Delta n_p^{\pm 2} \rangle_{g.c.e.} \equiv \langle \left(n_p^{\pm} \right)^2 \rangle_{g.c.e.} - \langle n_p^{\pm} \rangle_{g.c.e.}^2$$

$$= \langle n_p^{\pm} \rangle_{g.c.e.} \left(1 + \gamma \langle n_p^{\pm} \rangle_{g.c.e.} \right) \equiv v_p^{\pm\ 2}$$

$$\boxed{\langle \Delta n_p^{\alpha} \Delta n_k^{\beta} \rangle_{g.c.e.} = v_p^{\alpha\ 2}\ \delta_{p\,k}\ \delta_{\alpha\beta}}$$

$$\langle N_\alpha^2 \rangle_{g.c.e.} - \langle N_\alpha \rangle_{g.c.e.}^2 = \sum_{p,k} \langle n_p^\alpha n_k^\alpha \rangle_{g.c.e.}$$

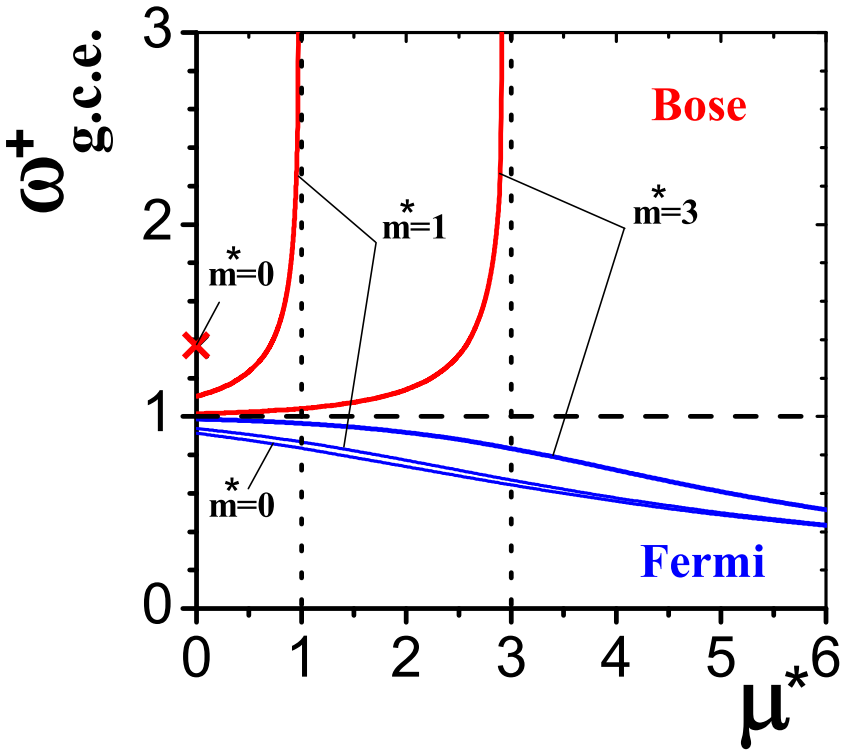
$$- \langle n_p^\alpha \rangle_{g.c.e.} \langle n_k^\alpha \rangle_{g.c.e.} = \sum_{p,k} \langle \Delta n_p^\alpha \Delta_k^\alpha \rangle_{g.c.e.} = \sum_p v_p^{\alpha\ 2}$$

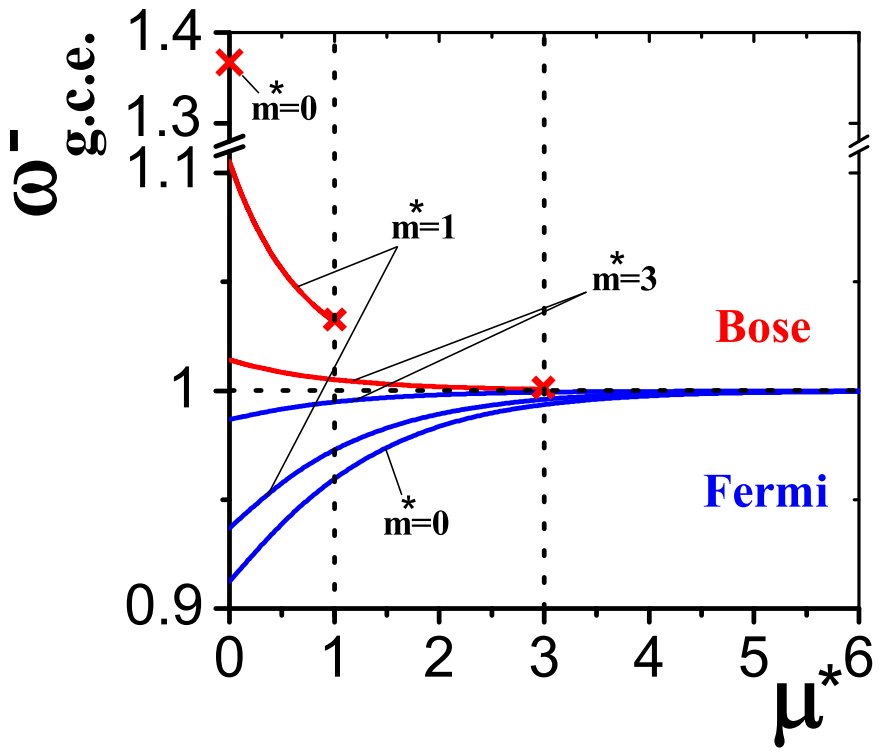
$$\omega^{\alpha}_{g.c.e.} \equiv \frac{\langle N^2_{\alpha}\rangle_{g.c.e.} - \langle N_{\alpha}\rangle^2_{g.c.e.}}{\langle N_{\alpha}\rangle_{g.c.e.}}$$

$$= \frac{\sum_{p,k}\langle \Delta n^{\alpha}_{p} \Delta n^{\alpha}_{k}\rangle_{g.c.e.}}{\sum_{p}\langle n^{\alpha}_{p}\rangle_{g.c.e.}} = \frac{\sum_{p} v^{\alpha}_{p}{}^{2}}{V\ \rho_{\alpha}}$$

$$\omega^{\alpha}_{g.c.e.} = 1 + \gamma \int_0^{\infty} \frac{x^2 dx}{\left[\exp\left(\sqrt{x^2 + m^{*2}} - \alpha\mu^*\right) - \gamma\right]^2}$$

$$\times \left[\int_0^{\infty} \frac{x^2 dx}{\exp\left(\sqrt{x^2 + m^{*2}} - \alpha\ \mu^*\right) - \gamma}\right]^{-1}$$


$\omega^+_{g.c.e.}$
Bose
Fermi
$m^*=0$
$m^*=1$
$m^*=3$
$m^*=0$
μ^*
0
1
2
3
4
5
6


$\omega^-_{g.c.e.}$
Bose
Fermi
$m^*=0$
$m^*=1$
$m^*=3$
$m^*=0$
μ^*
0.9
1
1.1
1.3
1.4
0
1
2
3
4
5
6

Particle number fluctuations in the CE

$$\{n_p^+, n_p^-\} \quad ?$$

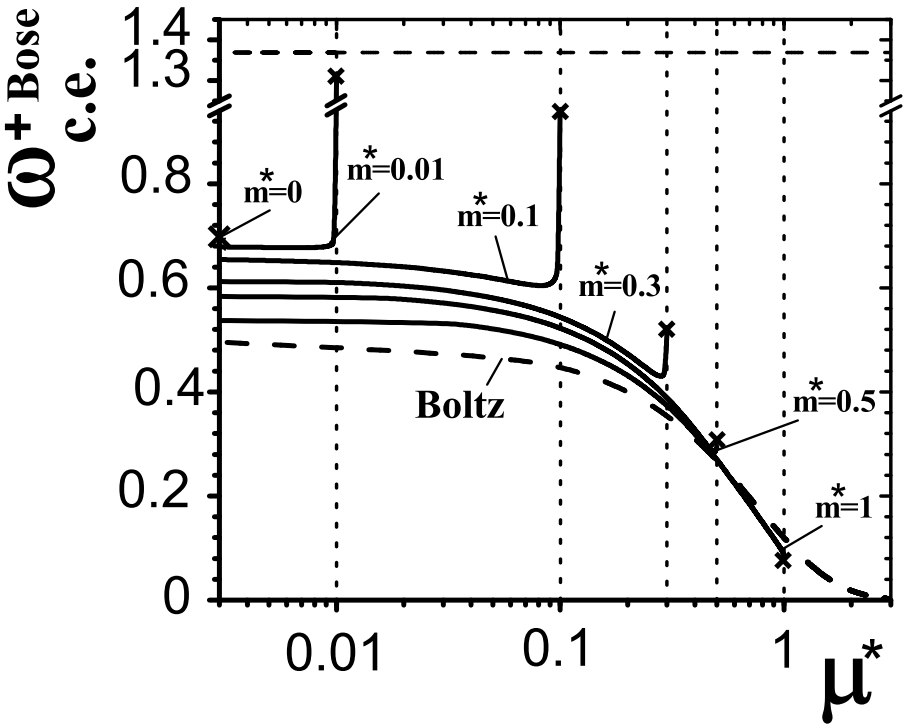
$$\Delta Q = \sum_p \left(\Delta n_p^+ \; - \; \Delta n_p^-\right) \; = \; 0$$

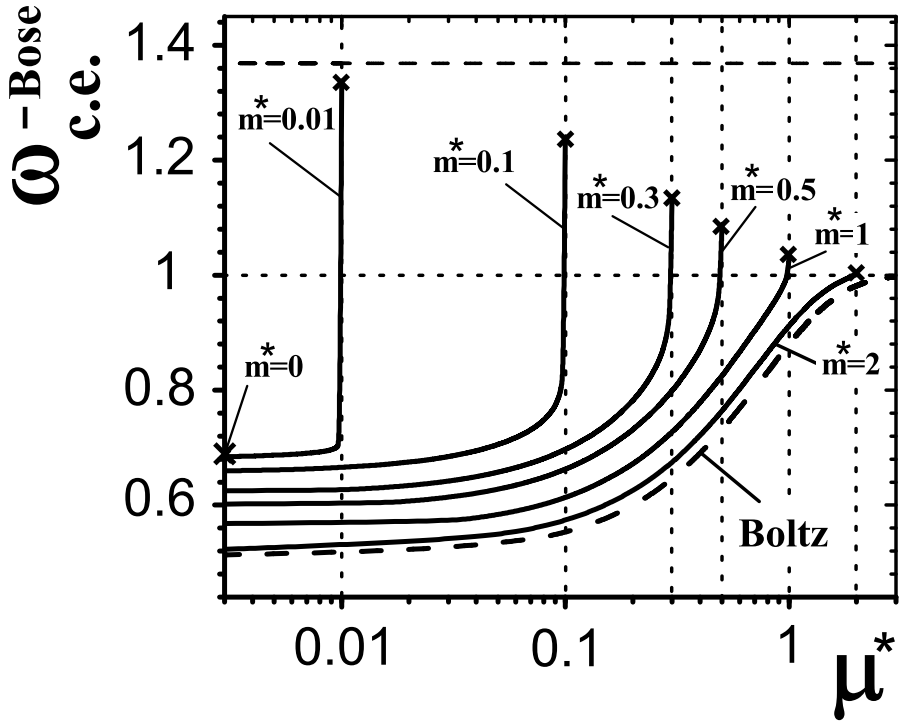
$$\langle Q \rangle_{g.c.e.} \; = \; Q \; , \qquad\qquad T, \; V$$

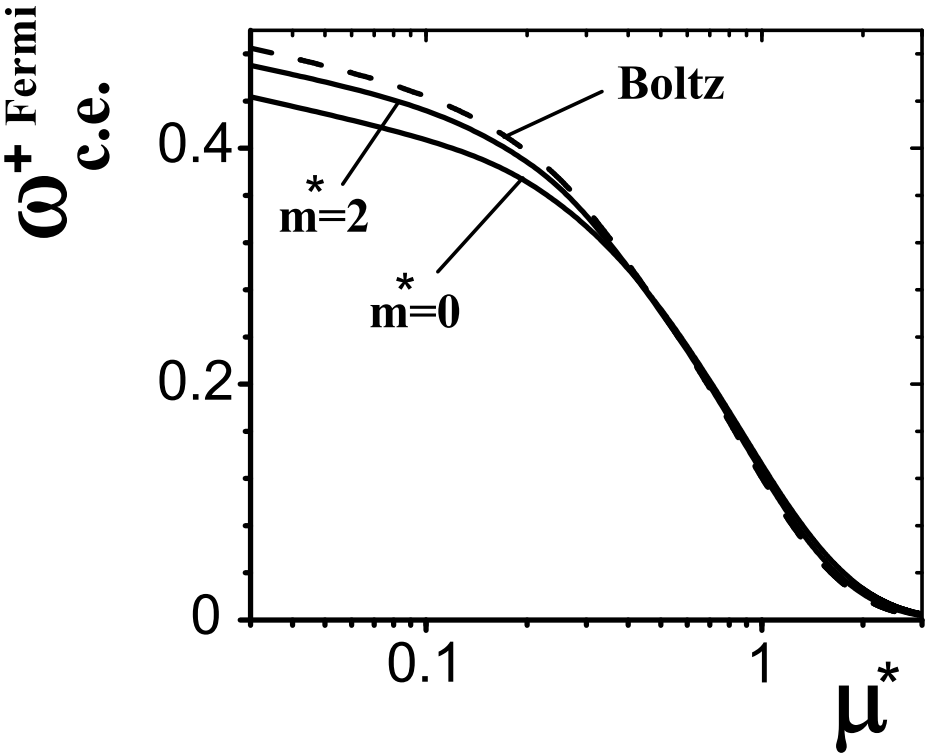
$$\langle N_+ \rangle_{c.e.} \; = \; \langle N_+ \rangle_{g.c.e.} \; , \qquad \langle N_- \rangle_{c.e.} \; = \; \langle N_- \rangle_{g.c.e.}$$

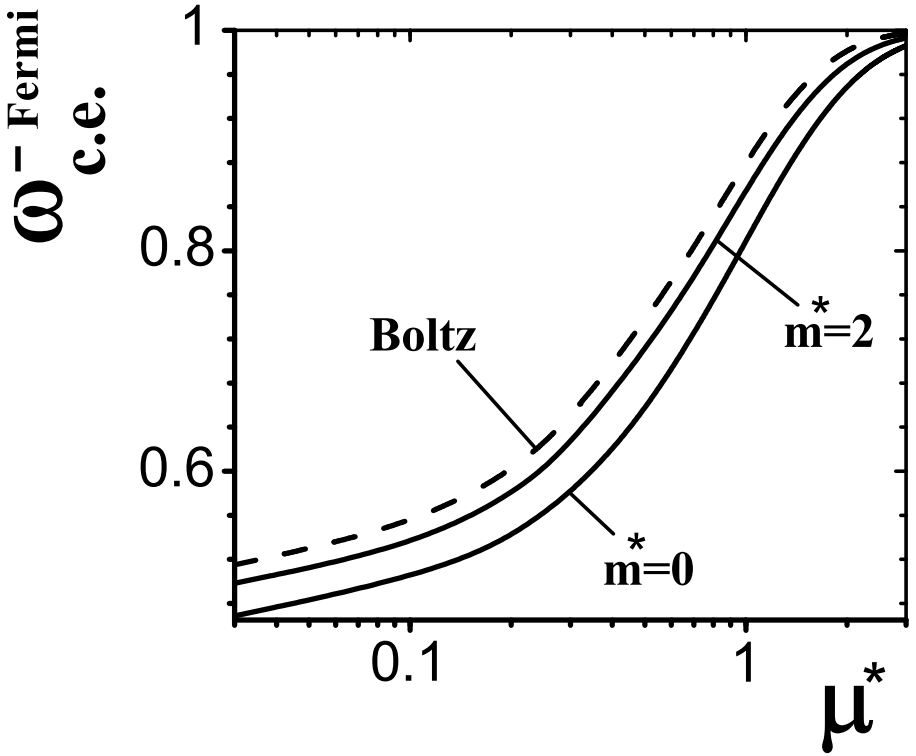
$$\langle \Delta n_p^\alpha \Delta n_k^\beta \rangle_{c.e.} = \delta_{p\,k} \; \delta_{\alpha\beta} \; {v_p^\alpha}^{\,2} \; - \; \frac{{v_p^\alpha}^{\,2} q^\alpha \; {v_k^\beta}^{\,2} q^\beta}{\sum_{p,\alpha} {v_p^\alpha}^{\,2}}$$

$$\omega_{c.e.}^\alpha \; = \; \frac{\sum_{p,k} \langle \Delta n_p^\alpha \Delta n_k^\alpha \rangle_{c.e.}}{\sum_p \langle n_p^\alpha \rangle_{c.e.}} \; = \frac{\sum_p {v_p^\alpha}^{\,2}}{V \; \rho_\alpha} \left(1 \; - \; \frac{\sum_p {v_p^\alpha}^{\,2}}{\sum_p {v_p^+}^{\,2} \; + \; \sum_p {v_p^-}^{\,2}}\right)$$

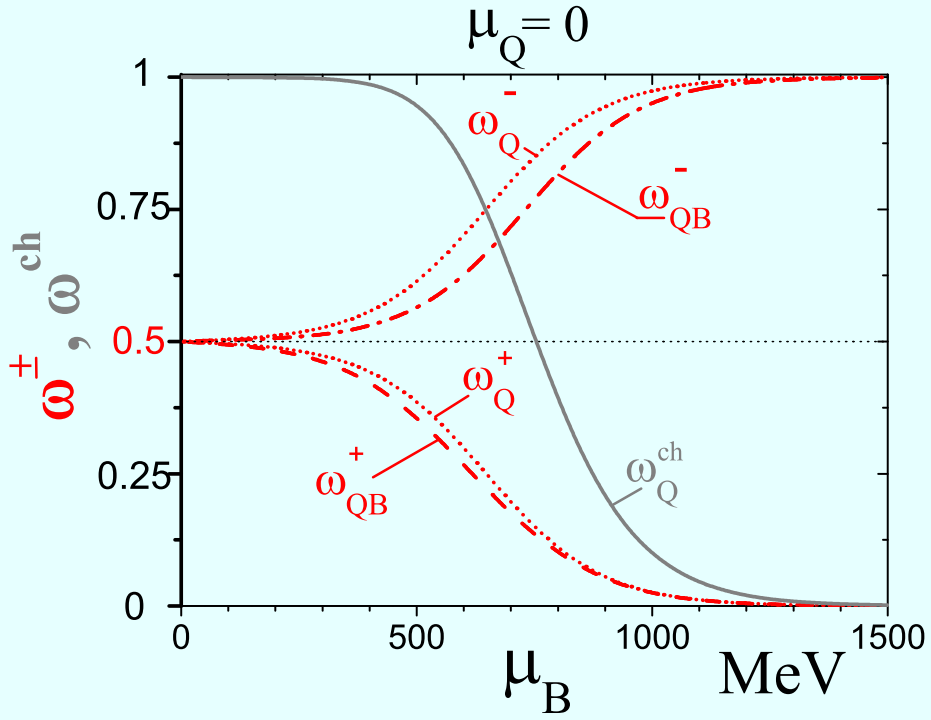

ω+ Bose c.e.
1.4
1.3
0.8
0.6
0.4
0.2
0
0.01
0.1
1
μ*
m*=0
m*=0.01
m*=0.1
m*=0.3
m*=0.5
m*=1
Boltz

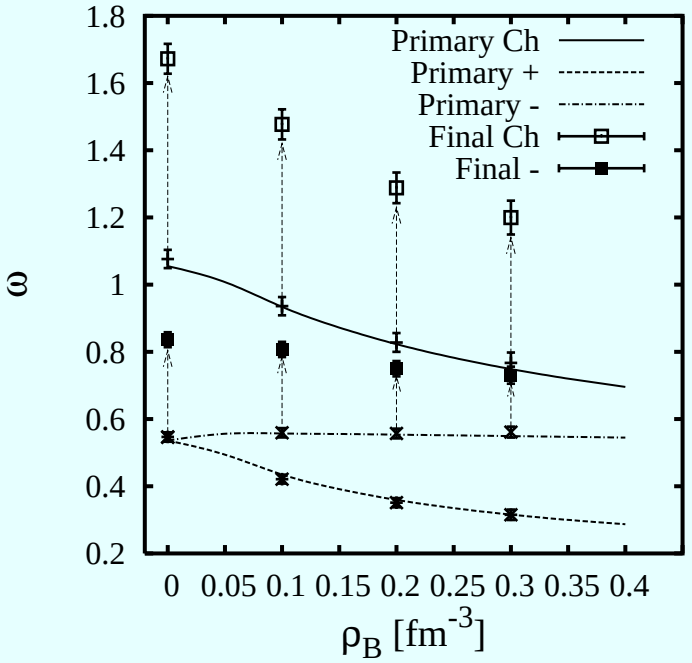

ω- Bose c.e.
1.4
1.2
1
0.8
0.6
0.01
0.1
1
μ*
m*=0.01
m*=0.1
m*=0.3
m*=0.5
m*=1
m*=2
m*=0
Boltz


ω^+ Fermi c.e.
0.4
0.2
0
0.1
1
μ^*
Boltz
m^*=2
m^*=0


ω^- Fermi c.e.
1
0.8
0.6
0.1
1
μ^*
Boltz
m^*=2
m^*=0

		$\mu = m = 0$	$\mu = m \to 0$	$\mu = m \to \infty$	$m \ll \mu \to \infty$
Grand Canonical Ensemble	$\omega_{g.c.e}^{+\ Boltz}$	1	1	1	1
	$\omega_{g.c.e}^{-\ Boltz}$	1	1	1	1
	$\omega_{g.c.e}^{+\ Bose}$	1.368	∞	∞	—
	$\omega_{g.c.e}^{-\ Bose}$	1.368	1.368	1	—
	$\omega_{g.c.e}^{+\ Fermi}$	0.912	0.912	0.791	0
	$\omega_{g.c.e}^{-\ Fermi}$	0.912	0.912	1	1
Canonical Ensemble	$\omega_{g.c.e}^{+\ Boltz}$	0.5	0.5	0	0
	$\omega_{g.c.e}^{-\ Boltz}$	0.5	0.5	1	1
	$\omega_{g.c.e}^{+\ Bose}$	0.684	1.368	0	—
	$\omega_{g.c.e}^{-\ Bose}$	0.684	1.368	1	—
	$\omega_{g.c.e}^{+\ Fermi}$	0.456	0.456	0	0
	$\omega_{g.c.e}^{-\ Fermi}$	0.456	0.456	1	1





$T = 120$ MeV, $Q \simeq 0.5$ B

V.V.Begun, et.al., Phys.Rev.C(2005).

$T = 160$ MeV, $Q = 0.4B$, $S = 0$.

A.Keranen, et. al., J.Phys.G (2005).

T [MeV]	120			160			180		
statistics	B	Q	Q GC	B	Q	Q GC	B	Q	Q GC
ω_B	0.500	0.500	1.000	0.500	0.500	1.000	0.500	0.500	1.000
ω_S	0.504	0.506	1.004	0.517	0.520	1.006	0.524	0.528	1.007
ω_-	0.502	0.535	1.066	0.509	0.536	1.055	0.512	0.535	1.045
$\omega_{\rm tot}$	1	1.061	1.061	1	1.047	1.047	1	1.038	1.038
ω_{π^-}	0.603	0.642	1.088	0.762	0.824	1.119	0.831	0.911	1.131
$\omega_{\rm p}$	0.892	0.892	1.000	0.931	0.931	0.999	0.942	0.941	0.998
ω_{K^+}	0.795	0.804	1.005	0.865	0.879	1.013	0.896	0.913	1.018

T [MeV]	120			160			180		
ρ_B [fm^{-3}]	0.0	0.15	0.30	0.0	0.15	0.30	0.0	0.15	0.30
ω_B	0.500	0.005	0.008	0.500	0.043	0.015	0.500	0.195	0.085
$\omega_{\bar{B}}$	0.500	1.000	1.000	0.500	0.952	0.980	0.500	0.799	0.905
ω_S	0.506	0.474	0.481	0.520	0.474	0.465	0.528	0.507	0.485
$\omega_{\bar{S}}$	0.506	0.493	0.494	0.520	0.493	0.486	0.528	0.514	0.498
ω_+	0.535	0.255	0.209	0.536	0.391	0.316	0.535	0.464	0.390
ω_-	0.535	0.571	0.566	0.536	0.555	0.549	0.535	0.562	0.558
ω_{ch}	1.066	0.649	0.543	1.055	0.872	0.749	1.045	0.974	0.872

Summary

I. $\omega_{g.c.e.}^{+\,Boltz} = \omega_{g.c.e.}^{-\,Boltz} = \omega_{g.c.e.}^{ch\,Boltz} = 1$

II. CE, MCE

a) Suppression of Fluctuations

b) dependance on μ

III. Bose and Fermi effects.

Outlook

$$P_{acc}(n,N) \;=\; q^n(1-q)^{N-n}\frac{N!}{n!(N-n)!}$$

$$\begin{aligned}
\langle n\rangle_{acc} &= \sum_{N=0}^{\infty} P(N) \sum_{n=0}^{N} nP_{acc}(n,N) \;=\; q\langle N\rangle \\
\langle n^2\rangle_{acc} &= \sum_{N=0}^{\infty} P(N) \sum_{n=0}^{N} n^2 P_{acc}(n,N) \\
&= q^2\langle N^2\rangle + q(1-q)\langle N\rangle
\end{aligned}$$

$$\langle N^k\rangle \;\equiv\; \sum_{N=0}^{\infty} N^k P(N)$$

$$\omega_{acc} \;=\; q\cdot\omega \;+\; (1-q)$$

$$P_{dyn}(N) \propto \exp[-\frac{(N-\overline{N})^2}{2\sigma^2}]$$

$$\langle N^2 \rangle - \langle N \rangle^2 = \sigma^2$$

$$\sigma = a\langle N \rangle$$

$$\omega = a^2\overline{N}$$

$$a = 0.1, \qquad \overline{N} \simeq 100$$